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# A Numerical Study of Couple System of Emden-Fowler Type Equations

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**ABSTRACT**

A dependable semi-analytical method via the application of a modified New Iterative Method (MNIM) to tackle the coupled system of Emden–Fowler-type equations has been proposed. More precisely, an effective integral operator in the sense of VIM was successfully constructed. Moreover, this operator is able to navigate to the closed-form solution easily without resorting to converting the coupled system to a system of integral equations; as in the case of a well-known reference in the literature. Lastly, the effectiveness of the method was demonstrated on some coupled systems of the governing model as well as other types of equations, and a speedier convergence rate was noted.

**Keywords:** New Iterative method, systems of nonlinear Emden-Fowler equation, Adomian's polynomials, and Lagrange's multiplier values.

## I. INTRODUCTION

Many scientific applications in the literature of mathematical physics and fluid mechanics can be distinctively described by the Emden–Fowler equation

$$y''(x)+\frac{k}{x}y'+f(x)g(y)=0, \qquad y(0)=y_0,$$
$$y'(0)=0 \qquad \ldots(1)$$

where $f(x)$ and $g(y)$ are some given functions of $x$ and $y$ respectively, and $k$ is called the shape factor. The Emden – Fowler Eqn. (1) describes a variety of phenomena in fluid mechanics, relativistic mechanics, pattern formation, population evolution and in chemically reacting systems. For $f(x)=1$ and $g(y)=y^m$, Eqn. (1) becomes the standard Lane–Emden equation of the first order and index $m$, given by

$$y''(x)+\frac{k}{x}y'+y^m=0,\ y(0)=y_0,$$
$$y'(0)=0 \qquad \ldots(2)$$

The Lane–Emden Eqn. (2) models the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules Khalique et al., (2009) and subject to the classical laws of thermodynamics. Moreover, the Lane–Emden equation of first order is a useful equation in astrophysics for computing the structure of interiors of polytropic stars. On the other side, for $f(x)=1$ and $g(y)=e^y$, Eqn. (1) becomes the standard Lane–Emden equation of the second order that models the non-dimensional density distribution $y(x)$ in an isothermal gas sphere

[ (Yousefi, 2006) and Khalique et al., (2009)] Moreover, the Lane–Emden equation (2) describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of thermodynamics. In addition, the Lane-Emden equation of the first kind appears also in other context such as in the case of radiatively cooling, self-gravitating gas clouds, in the mean-field treatment of a phase transition in critical adsorption and in the modeling of clusters of galaxies [(Polat & Polat, 2010), (Eggleton, 2011) and (A. Wazwaz, 2011)].

The Lane-Emden equation was first studied by astrophysicists Jonathan Homer Lane and Robert Emden, where they considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. The well-known Lane-Emden equation has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, the theory of thermionic currents, and in the modeling of clusters of galaxies. The Emden–Fowler equation was studied by Fowler in Fowler (1931) to describe a variety of phenomena in fluid mechanics and relativistic mechanics among others. The singular behavior that occurs at x = 0 and the emergence of unwanted terms from the use of some semi-analytical methods is the main difficulty of Eqn. (1) and (2) [ Asadpour et al., (2019), Öztürk*, (2019), Al-Jawary & Al-Qaissy, (2015) and Wazwaz et al., (2013) and Alsulami et al., (2022)].

The motivation for the analysis that we will present in this seminar comes actually from the aim of introducing a reliable framework depending mainly on the New Iterative Method (NIM) [Adwan & Radhi, (2020), Hemeda, (2018), Jajarmi & Baleanu, (2020) and ( Liu et al., 2021) ] to handle this type of equations. In recent years a lot of attention has been devoted to the study of New Iterative Method (NIM) to investigate various scientific models. The New Iterative Method (NIM), which accurately computes the series solution, is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that are elegantly computed. The main advantage of the method is that it can be applied directly for all types of differential and integral equations. Another important advantage is that the method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution.

A major drawback of the NIM is how this method can be modified to handle the concept of singular points. To properly address this problem, we may require slight modification of the NIM algorithm as described in Hemeda, (2018). In this work we establish an alternative framework by modifying the New Iterative Method (NIM) to accommodate the couple system of Emden-Fowler equation.

## II. THE MODIFIED NEW ITERATIVE METHOD FOR THE SOLUTION OF THE EMDEN-FOWLER EQUATION

Singular initial value problems in the second order ordinary differential equations occur in several models of mathematical physics and astrophysics (Vedat, 2007), such as the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres and theory of thermionic currents which are modeled by means of the following Emden-Fowler equation:

$$u''(x)+\frac{\alpha}{x}u'(x)+u^{m}(x)=g(x), \quad 0<x\le 1, \quad \alpha>0, \quad m>0. \quad \ldots(3)$$

Under the following initial condition

$$u(0)=A,\ u'(0)=B \quad \ldots(4)$$

Where A and B are constants, $u^{m}(x)$ is a continuous real valued function and $g(x)\in[0,1]$. Now, in order to overcome the singularity of Eqn. (3) without any loss of generality, we rewrite Eqn. (3) in the correctional form offered in (A. Wazwaz et al., 2013) as follows:

$$u(x)=A+Bx+\int_0^x\left(\frac{t^2-xt}{x}\right)\left[u''(t)+\frac{\alpha}{t}u'(t)+u^{m}(t)-g(t)\right]dt \quad \ldots(5)$$

To use the MNIM; we let

$$u(x)=\sum_{n=0}^{\infty}u_n(x) \quad \ldots(6)$$

It was observed that the computation of each component $u_n(x), n\ge 1$ and $m>1$ results in some unwanted terms which are the major short comings of the traditional NIM. So in this modification we shall be considering a new formation that will be voids of the unwanted terms. The formation is as follows:
Substituting Eqn. (6) in Eqn. (5) and splitting the resulting equation, we get the following relation owing to the traditional NIM.

$$u_0(x) = A + Bx$$
...(7)

$$u_1(x) = \int_0^x \left( \frac{t^2 - xt}{x} \right) \left[ u_0''(t) + \frac{\alpha}{t} u_0'(t) + u_0{}^m(t) - g(t) \right] dt$$
...(8)

$$u_2(x) = \int_0^x \left( \frac{t^2 - xt}{x} \right) \left[ \sum_{n=0}^{1} u_n''(t) + \frac{\alpha}{t} \sum_{n=0}^{1} u_n'(t) + \sum_{n=0}^{1} u_n{}^m(t) - g(t) \right] dt$$
...(9)

$$u_3(x) = \int_0^x \left( \frac{t^2 - xt}{x} \right) \left[ \sum_{n=0}^{2} u_n''(t) + \frac{\alpha}{t} \sum_{n=0}^{2} u_n'(t) + \sum_{n=0}^{2} u_n{}^m(t) - g(t) \right] dt$$
...(10)

$$\vdots$$

$$u_{n+1}(x) = \int_0^x \left( \frac{t^2 - xt}{x} \right) \left[ \sum_{n=0}^{n} u_n''(t) + \frac{\alpha}{t} \sum_{n=0}^{n} u_n'(t) + \sum_{n=0}^{n} u_n{}^m(t) - g(t) \right] dt$$
...(11)

## III. THE CASE OF COUPLED SYSTEM OF EMDEN–FOWLER TYPE EQUATIONS

This section gives a mathematical algorithm for the determination of recursive solution of the inhomogeneous coupled system of Emden–Fowler type equations. This algorithm is based upon the modified NIM procedure for the solution of various functional equations.

Let us consider the inhomogeneous coupled system of Emden–Fowler type equations

$$\begin{cases} y_1''(x) + \dfrac{k_1}{x} y_1' + f_1(y_1, y_2) = g_1(x), \\ y_2''(x) + \dfrac{k_2}{x} y_2' + f_2(y_1, y_2) = g_2(x) \end{cases} \quad ...(12)$$

With initial condition

$$\begin{cases} y_1(0) = c_0, y_1'(0) = c_1, \\ y_2(0) = d_0, y_2'(0) = d_1. \end{cases} \quad ...(13)$$

where $k_1 > 0$, $k_2 > 0$ are real constants, and $g_1(x)$ and $g_2(x)$ are given functions of $x$; while $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ are prescribed analytic nonlinear functions of $y_1$ and $y_2$. This equation arises in certain physical processes, such as population growth, pattern formation, and chemical reaction, among others. Now, in order to overcome the singularity of Eqn. (3) without any loss of generality, we rewrite Eqn. (3) in the correctional form offered in (A. Wazwaz et al., 2013) as follows:

$$\begin{cases} y_1(x) = c_0 + c_1 x + \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ y_1''(t) + \frac{k_1}{x} y_1'(t) + f_1(y_1, y_2) - g_1(t)\right] dt, \\ y_2(x) = d_0 + d_1 x + \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ y_2''(t) + \frac{k_2}{x} y_2'(t) + f_2(y_1, y_2) - g_2(t)\right] dt \end{cases} \quad \ldots(14)$$

To use the MNIM; we let

$$\begin{cases} y_1(x) = \sum_{n=0}^{\infty} y_{1,n}(x), \\ y_2(x) = \sum_{n=0}^{\infty} y_{2,n}(x). \end{cases} \quad \ldots(15)$$

It was observed that the computation of each component $u_n(x), n \geq 1$ and $m > 1$ results in some unwanted terms which are the major short comings of the traditional NIM. So in this modification we shall be considering a new formation that will be voids of the unwanted terms. The formation is as follows:

Substituting Eqn. (15) in Eqn. (14) and splitting the resulting equation, we get the following relation owing to the NIM.

$$\begin{cases} y_{1,0}(x) = c_0 + c_1 x, \\ y_{2,0}(x) = d_0 + d_1 x. \end{cases} \quad \ldots(16)$$

$$\begin{cases} y_{1,1}(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ y_{1,0}''(t) + \frac{k_1}{x} y_{1,0}'(t) + f_{1,0}(y_{1,0}, y_{2,0}) - g_1(t)\right] dt, \\ y_{2,1}(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ y_{2,0}''(t) + \frac{k_2}{x} y_{2,0}'(t) + f_{2,0}(y_{1,0}, y_{2,0}) - g_2(t)\right] dt. \end{cases} \quad \ldots(17)$$

$$\begin{cases} y_{1,2}(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ \sum_{n=0}^{1} y_{1,n}''(t) + \frac{k_1}{t} \sum_{n=0}^{1} y_{1,n}'(t) + \sum_{n=0}^{1} f_{1,0}(y_{1,0}, y_{2,0}) - g_1(t)\right] dt, \\ y_{2,2}(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ \sum_{n=0}^{1} y_{2,n}''(t) + \frac{k_2}{t} \sum_{n=0}^{1} y_{2,n}'(t) + \sum_{n=0}^{1} f_{2,0}(y_{1,0}, y_{2,0}) - g_2(t)\right] dt. \end{cases}$$

$\ldots(18)$

$$\begin{cases} y_{1,3}(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ \sum_{n=0}^{2} y_{1,n}''(t) + \frac{k_1}{t} \sum_{n=0}^{2} y_{1,n}'(t) + \sum_{n=0}^{2} f_{1,0}(y_{1,0}, y_{2,0}) - g_1(t)\right] dt, \\ y_{2,3}(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ \sum_{n=0}^{2} y_{2,n}''(t) + \frac{k_2}{t} \sum_{n=0}^{2} y_{2,n}'(t) + \sum_{n=0}^{2} f_{2,0}(y_{1,0}, y_{2,0}) - g_2(t)\right] dt. \end{cases}$$

$\ldots(19)$

$\vdots$

And so on. Continuing in this manner, the $(n+1)^{th}$ approximation of the exact solutions for the unknown functions $y_1(x)$ and $y_2(x)$ can be achieved as

$$\begin{cases} y_{1,n+1}(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ \sum_{m=0}^{n} y''_{1,m}(t) + \frac{k_1}{t}\sum_{m=0}^{n} y'_{1,m}(t) + \sum_{m=0}^{n} f_{1,m}(y_{1,m}, y_{2,m}) - g_1(t)\right]dt, \\ y_{2,n+1}(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[ \sum_{m=0}^{n} y''_{2,m}(t) + \frac{k_2}{t}\sum_{m=0}^{n} y'_{2,m}(t) + \sum_{m=0}^{n} f_{2,m}(y_{1,m}, y_{2,m}) - g_2(t)\right]dt. \end{cases}$$

…(20)

Therefore, the approximate solutions

$$\begin{cases} y_1(x) = \sum_{m=0}^{n+1} y_{1,m}(x), \\ y_2(x) = \sum_{m=0}^{n+1} y_{2,m}(x). \end{cases} \quad \ldots(21)$$

The modified NIM will be illustrated by discussing some problems on the couple systems of linear Emden-Fowler Equations, nonlinear couple systems of Emden-Fowler Equations, the linear Lane-Emden Equation and nonlinear lane-Emden Equation.

## IV. RESULTS

The present section examines the application of the proposed algorithms on different test singular problems of the coupled system of Emden–Fowler-type equations and other types of differential equations.

## V. LINEAR SYSTEMS OF EMDEN–FOWLER EQUATION

Here, we will study two linear systems Emden–Fowler equation with singular behaviour at $x = 0$. The two initial value problems are selected for a variety of values of $\alpha_i$ .

**Example 1:** Consider the coupled system of Emden-Fowler type equations as follows [ Alsulami et al., (2022)]

$$\begin{cases} u''(x) + \frac{3}{x}u'(x) - 4[u(x) + v(x)] = 0, \\ v''(x) + \frac{2}{x}v'(x) + 3[u(x) + v(x)] = 0, \end{cases} \quad \ldots(22)$$

with initial conditions

$$\begin{cases} u(0) = 1 = v(0), \\ u'(0) = 0 = v'(0). \end{cases} \quad \ldots(23)$$

The exact solution for the system is

$$\begin{cases} u(x) = 1 + x^2, \\ v(x) = 1 - x^2. \end{cases} \quad \ldots(24)$$

The algorithm in section (**3.0**), yields the following components for Eqn. (22)

$$\begin{cases} u(x) = 1 + 0x + \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[u''(t) + \frac{3}{t}u'(t) - 4[u(t) + v(t)]\right]dt, \\ v(x) = 1 + 0x + \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[v''(t) + \frac{2}{t}v'(t) + 3[u(t) + v(t)]\right]dt \end{cases} \quad \ldots(25)$$

$$\begin{cases} u_0(x) = 1, \\ v_0(x) = 1 \end{cases} \quad \ldots 26)$$

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$$\begin{cases} u_1(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[u_0''(t)+\frac{3}{t}u_0'(t)-4[u_0(t)+v_0(t)]\right]=x^2 \\ v_1(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[v_0''(t)+\frac{2}{t}v_0'(t)+3[u_0(t)+v_0(t)]\right]=-x^2 \end{cases} \quad \ldots(27)$$

$$\begin{cases} u_2(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{1}u_n''(t)+\frac{3}{t}\sum_{n=0}^{1}u_n'(t)-4\sum_{n=0}^{1}[u_n(t)+v_n(t)]\right]=0 \\ v_2(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{1}v_n''(t)+\frac{2}{t}\sum_{n=0}^{1}v_n'(t)+3\sum_{n=0}^{1}[u_n(t)+v_n(t)]\right]=0 \end{cases} \quad \ldots(28)$$

$$\begin{cases} u_3(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{2}u_n''(t)+\frac{3}{t}\sum_{n=0}^{2}u_n'(t)-4\sum_{n=0}^{2}[u_n(t)+v_n(t)]\right]=0 \\ v_3(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{2}v_n''(t)+\frac{2}{t}\sum_{n=0}^{2}v_n'(t)+3\sum_{n=0}^{2}[u_n(t)+v_n(t)]\right]=0 \end{cases} \quad \ldots(29)$$

$\vdots$

The series solution is then obtain by summing the above iterations

$$\begin{cases} u(x)=u_0(x)+u_1(x)+u_2(x)+u_3(x)+\ldots=1+x^2+0+0+\ldots \\ v(x)=v_0(x)+v_1(x)+v_2(x)+v_3(x)+\ldots=1-x^2+0+0+\ldots \end{cases} \quad \ldots(30)$$

This gives the exact solution of Eqn. (22) which is given by

$$\begin{cases} u(x)=1+x^2, \\ v(x)=1-x^2. \end{cases} \quad \ldots(31)$$

**Example 2:** We study the system of linear equations of Emden–Fowler type [Wazwaz, (2011)]

$$\begin{cases} u''(x)+\frac{1}{x}u'(x)+v(x)=x^3+5, \\ v''(x)+\frac{2}{x}v'(x)+w(x)=x^4+12x+1, \\ w'''(x)+\frac{3}{x}w'(x)+u(x)=25x^2+1, \end{cases} \quad \ldots(32)$$

with initial conditions

$$\begin{cases} u(0)=v(0)=w(0)=1, \\ u'(0)=v'(0)=w'(0)=0, \end{cases} \quad \ldots(33)$$

The exact solution for the system is

$$\begin{cases} u(x)=1+x^2, \\ v(x)=1+x^3, \\ w(x)=1+x^4. \end{cases} \quad \ldots(34)$$

The algorithm in section (**3.0**), yields the following components for Eqn. (32)

$$\begin{cases} u(x)=1+0x+\int_0^x\left(\frac{t^2-xt}{x}\right)\left[u''(t)+\frac{1}{t}u'(t)+v(t)-t^3-5\right]dt\,, \\ v(x)=1+0x+\int_0^x\left(\frac{t^2-xt}{x}\right)\left[v''(t)+\frac{2}{t}v'(t)+w(t)-t^4-12t-1\right]dt \\ w(x)=1+0x+\int_0^x\left(\frac{t^2-xt}{x}\right)\left[w''(t)+\frac{3}{t}w'(t)+u(t)-25t^2-1\right]dt \end{cases} \quad \ldots(35)$$

$$\begin{cases} u_0(x)=1, \\ v_0(x)=1, \\ w_0(x)=1. \end{cases} \quad \ldots(36)$$

$$\begin{cases} u_1(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[u_0''(t)+\frac{1}{t}u_0'(t)+v_0(t)-t^3-5\right]dt=\frac{1}{25}x^5+x^2 \\ v_1(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[v_0''(t)+\frac{2}{t}v_0'(t)+w_0(t)-t^4-12t-1\right]dt=\frac{1}{42}x^6+x^3 \\ w_1(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[w''(t)+\frac{3}{t}w_0'(t)+u_0(t)-25t^2-1\right]dt=\frac{25}{24}x^4 \end{cases} \quad \ldots(37)$$

$$\begin{cases} u_2(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{1}u_n''(t)+\frac{1}{t}\sum_{n=0}^{1}u_n'(t)+\sum_{n=0}^{1}v_n(t)-t^3-5\right]dt=-\frac{1}{2688}x^8-\frac{1}{25}x^5 \\ v_2(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{1}v_n''(t)+\frac{2}{t}\sum_{n=0}^{1}v_n'(t)+\sum_{n=0}^{1}w_n(t)-t^4-12t-1\right]dt=-\frac{25}{1008}x^6 \\ w_2(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{1}w_n''(t)+\frac{3}{t}\sum_{n=0}^{1}w_n'(t)+\sum_{n=0}^{1}u_n'(t)-25t^2-1\right]dt=-\frac{1}{1575}x^7-\frac{1}{24}x^4 \end{cases}$$

$\ldots(38)$

$$\begin{cases} u_3(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{2}u_n''(t)+\frac{1}{t}\sum_{n=0}^{2}u_n'(t)+\sum_{n=0}^{2}v_n(t)-t^3-5\right]dt=\frac{25}{64512}x^8 \\ v_3(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{2}v_n''(t)+\frac{2}{t}\sum_{n=0}^{2}v_n'(t)+\sum_{n=0}^{2}w_n(t)-t^4-12t-1\right]dt=\frac{1}{141750}x^9+\frac{1}{1008}x^6 \\ w_3(x)=\int_0^x\left(\frac{t^2-xt}{x}\right)\left[\sum_{n=0}^{2}w_n''(t)+\frac{3}{t}\sum_{n=0}^{2}w_n'(t)+\sum_{n=0}^{2}u_n'(t)-25t^2-1\right]dt=\frac{1}{1575}x^7+\frac{1}{322560}x^{10} \end{cases}$$

$\ldots(39)$

$$\begin{cases} u_4(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[\sum_{n=0}^{3} u_n''(t) + \frac{1}{t}\sum_{n=0}^{3} u_n'(t) + \sum_{n=0}^{3} v_n(t) - t^3 - 5\right]dt = \\ -\frac{1}{17151750}x^{11} - \frac{1}{64512}x^8 \\ v_4(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[\sum_{n=0}^{3} v_n''(t) + \frac{2}{t}\sum_{n=0}^{3} v_n'(t) + \sum_{n=0}^{3} w_n(t) - t^4 - 12t - 1\right]dt = \\ -\frac{1}{50319360}x^{12} - \frac{1}{141750}x^9 \\ w_4(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[\sum_{n=0}^{3} w_n''(t) + \frac{3}{t}\sum_{n=0}^{3} w_n'(t) + \sum_{n=0}^{3} u_n'(t) - 25t^2 - 1\right]dt = \\ -\frac{5}{1548288}x^{10} \end{cases} \quad \ldots(40)$$

$\vdots$

The series solution is then obtain by summing the above iterations

$$\begin{cases} u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \ldots = 1 + x^2 - \frac{1}{17151750}x^{11}\ldots \\ v(x) = v_0(x) + v_1(x) + v_2(x) + v_3(x) + v_4(x) + \ldots = 1 + x^3 - \frac{1}{5019360}x^{12}\ldots \\ w(x) = w_0(x) + w_1(x) + w_2(x) + w_3(x) + w_4(x) + \ldots = 1 + x^4 - \frac{1}{7741440}x^{10}\ldots \end{cases}$$

$\ldots(41)$

Neglecting the noise terms, we get the solutions in closed form as;

$$\begin{cases} u(x) = 1 + x^2, \\ v(x) = 1 + x^3, \\ w(x) = 1 + x^4. \end{cases} \quad \ldots(42)$$

**Table 1: Numerical Results for Approximate Solution of $u(x)$ in Example 2.**

| x | Exact Solution u(x) | MNIM u(x) | NIM u(x) |
|---|---|---|---|
| **0** | 1.00000 | 1.00000 | 1.00000 |
| **0.1** | 1.01000 | 1.01000 | 1.01000 |
| **0.2** | 1.04000 | 1.04000 | 1.04000 |
| **0.3** | 1.09000 | 1.09000 | 1.09000 |
| **0.4** | 1.16000 | 1.16000 | 1.16000 |
| **0.5** | 1.25000 | 1.25000 | 1.25000 |
| **0.6** | 1.36000 | 1.36000 | 1.36001 |
| **0.7** | 1.49000 | 1.49000 | 1.49002 |
| **0.8** | 1.64000 | 1.64000 | 1.64007 |
| **0.9** | 1.81000 | 1.81000 | 1.81017 |

| | | | |
|---|---|---|---|
| **1** | 2.00000 | 2.00000 | 2.00039 |

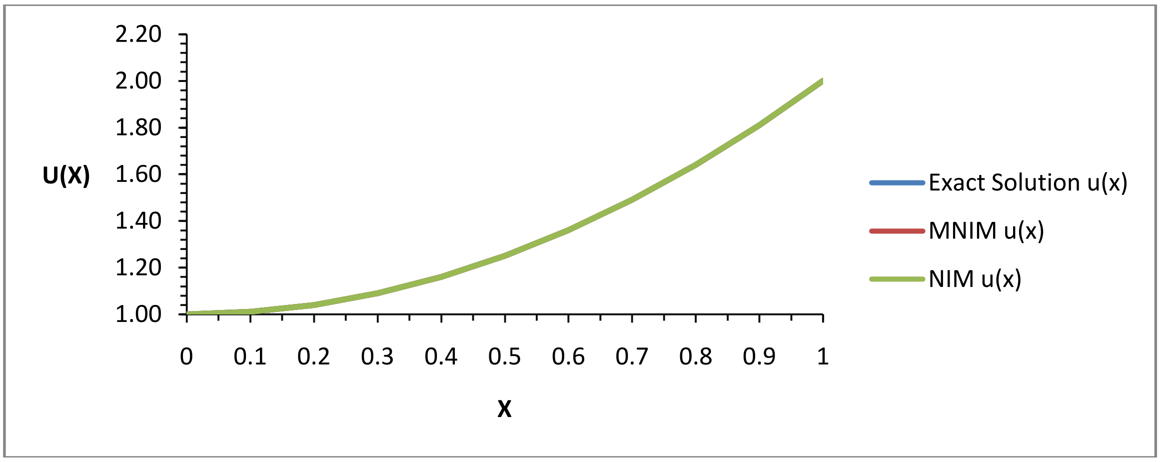


Figure 1: Plots of the exact solution $u(x)$ , NIM and the approximate solution using MNIM.

**Table 2: Numerical Results for Approximate Solution of $v(x)$ in Example 2.**

| **x** | **Exact Solution v(x)** | **MNIM v(x)** | **NIM v(x)** | **MNIM Error** | **NIM Error** |
|---|---|---|---|---|---|
| **0** | 1.00000 | 1.00000 | 1.00000 | 0 | 0 |
| **0.1** | 1.00000 | 1.00100 | 1.00100 | 0 | 2.48E-08 |
| **0.2** | 1.01000 | 1.00800 | 1.00800 | 0 | 1.587E-06 |
| **0.3** | 1.03000 | 1.02700 | 1.02698 | 1.0658E-14 | 1.808E-05 |
| **0.4** | 1.06000 | 1.06400 | 1.06390 | 3.3351E-13 | 0.0001016 |
| **0.5** | 1.13000 | 1.12500 | 1.12461 | 4.8519E-12 | 0.0003875 |
| **0.6** | 1.22000 | 1.21600 | 1.21484 | 4.3259E-11 | 0.0011571 |
| **0.7** | 1.34000 | 1.34300 | 1.34008 | 2.7507E-10 | 0.0029176 |
| **0.8** | 1.51000 | 1.51200 | 1.50550 | 1.3657E-09 | 0.0065006 |
| **0.9** | 1.73000 | 1.72900 | 1.71582 | 5.6127E-09 | 0.0131778 |
| **1** | 2.00000 | 2.00000 | 1.97521 | 1.9873E-08 | 0.0247945 |

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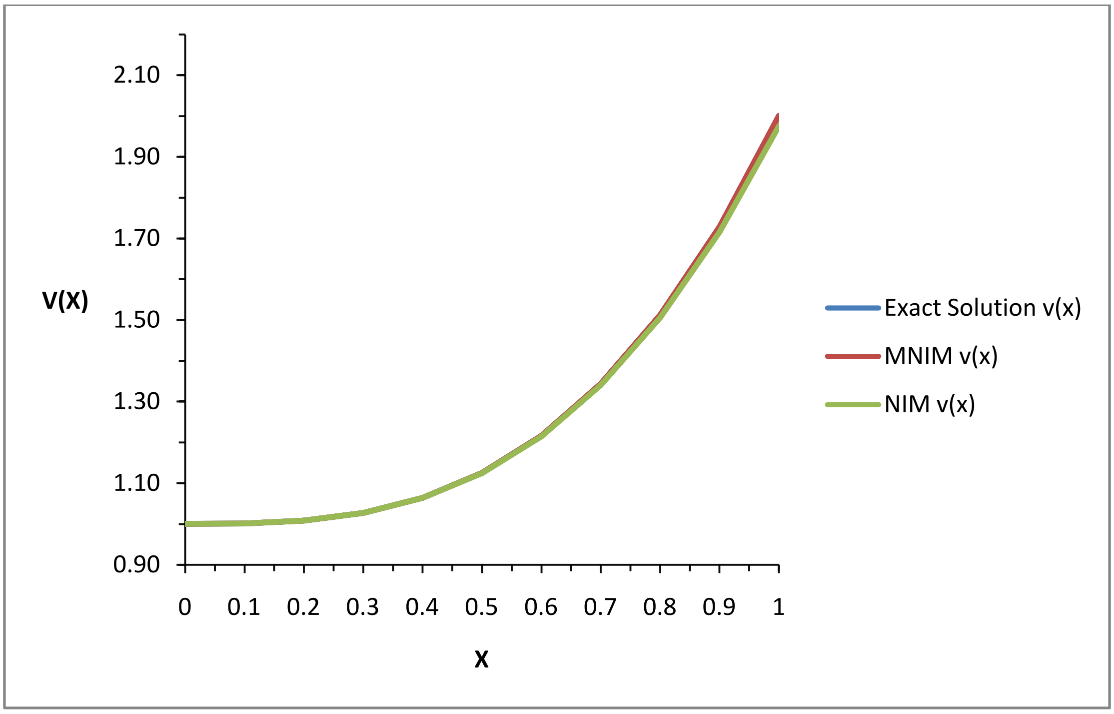


Figure 2: Plots of the exact solution $v(x)$ , NIM and the approximate solution using MNIM.

**Table 3: Numerical Results for Approximate Solution of $w(x)$ in Example 2.**

| x | Exact Solution w(x) | MNIM w(x) | NIM w(x) | NIM Error |
|---|---|---|---|---|
| **0** | 1.00000 | 1.00000 | 1.00000 | 0 |
| **0.1** | 1.00010 | 1.00010 | 1.00010 | 1.5543E-14 |
| **0.2** | 1.00160 | 1.00160 | 1.00160 | 1.5873E-11 |
| **0.3** | 1.00810 | 1.00810 | 1.00810 | 9.1532E-10 |
| **0.4** | 1.02560 | 1.02560 | 1.02560 | 1.6254E-08 |
| **0.5** | 1.06250 | 1.06250 | 1.06250 | 1.5138E-07 |
| **0.6** | 1.12960 | 1.12960 | 1.12960 | 9.3729E-07 |
| **0.7** | 1.24010 | 1.24010 | 1.24010 | 4.3786E-06 |
| **0.8** | 1.40960 | 1.40960 | 1.40962 | 1.6644E-05 |
| **0.9** | 1.65610 | 1.65610 | 1.65615 | 5.4049E-05 |
| **1** | 2.00000 | 2.00000 | 2.00016 | 0.00015501 |

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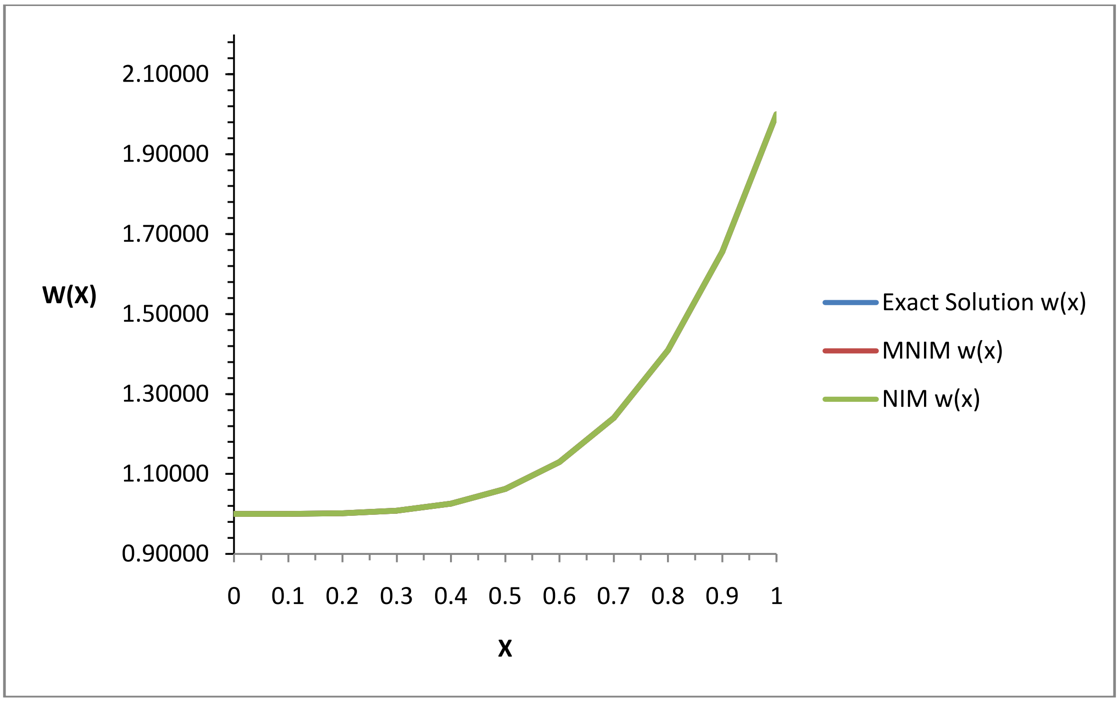


Figure 3: Plots of the exact solution $w(x)$, NIM and the approximate solution using MNIM.

## VI. NONLINEAR SYSTEMS OF EMDEN–FOWLER EQUATION

Here, we will study two nonlinear systems Emden–Fowler equation.

**Example 3:** We next study the system of nonlinear equations of Lane–Emden-type [ Alsulami et al., (2022), Wazwaz, (2011) and Öztürk*, (2019)]

$$\begin{cases} u''(x)+\dfrac{2}{x}u'(x)+v^2(x)-u^2(x)+6v(x)=6+6x^2, \\ v''(x)+\dfrac{2}{x}v'(x)+u^2(x)-v^2(x)-6v(x)=6-6x^2, \end{cases} \quad \ldots(43)$$

with initial conditions

$$\begin{cases} u(0)=1, v(0)=-1, \\ u'(0)=v'(0)=0. \end{cases} \quad \ldots(44)$$

The exact solution for the system is

$$\begin{cases} u(x)=x^2+e^{x^2}, \\ v(x)=x^2-e^{x^2}. \end{cases} \quad \ldots(45)$$

The algorithm in section (**3.0**), yields the following components for Eqn. (43)

$$\begin{cases} u(x)=1+0x+\displaystyle\int_0^x \left(\frac{t^2-xt}{x}\right)\left[u''(t)+\frac{2}{t}u'(t)+v^2(t)-u^2(t)+6v(t)-6-6t^2\right]dt, \\ v(x)=-1+0x+\displaystyle\int_0^x \left(\frac{t^2-xt}{x}\right)\left[v''(t)+\frac{2}{t}v'(t)+u^2(t)-v^2(t)-6v(t)-6+6t^2\right]dt \end{cases} \quad \ldots(46)$$

$$\begin{cases} u_0(x)=1, \\ v_0(x)=-1 \end{cases} \quad \ldots(47)$$

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$$\begin{cases} u_1(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[u_0''(t) + \frac{2}{t}u_0'(t) + v_0^2(t) - u_0^2(t) + 6v_0(t) - 6 - 6t^2\right]dt = \frac{3}{10}x^4 + 2x^2 \\ v_1(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[v_0''(t) + \frac{2}{t}v_0'(t) + u_0^2(t) - v_0^2(t) - 6v_0(t) - 6 + 6t^2\right]dt = -\frac{3}{10}x^4 \end{cases} \quad \ldots(48)$$

$$\begin{cases} u_2(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[\sum_{n=0}^{1} u_n''(t) + \frac{2}{t}\sum_{n=0}^{1} u_n'(t) + \sum_{n=0}^{1} v_n^2(t) - \sum_{n=0}^{1} u_n^2(t) + 6\sum_{n=0}^{1} v_n(t) - 6 - 6t^2\right]dt = \\ \frac{1}{60}x^8 + \frac{29}{210}x^6 + \frac{1}{5}x^4 \\ v_2(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[\sum_{n=0}^{1} v_n''(t) + \frac{2}{t}\sum_{n=0}^{1} v_n'(t) + \sum_{n=0}^{1} u_n^2(t) - \sum_{n=0}^{1} v_n^2(t) - 6\sum_{n=0}^{1} v_n(t) - 6 + 6t^2\right]dt = \\ -\frac{1}{60}x^8 - \frac{29}{210}x^6 - \frac{1}{5}x^4 \end{cases}$$

$\ldots(49)$

$$\begin{cases} u_3(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[\sum_{n=0}^{2} u_n''(t) + \frac{2}{t}\sum_{n=0}^{2} u_n'(t) + \sum_{n=0}^{2} v_n^2(t) - \sum_{n=0}^{2} u_n^2(t) + 6\sum_{n=0}^{2} v_n(t) - 6 - 6t^2\right]dt = \\ \frac{1}{2340}x^{12} + \frac{137}{23100}x^{10} + \frac{19}{840}x^8 + \frac{1}{35}x^6 \\ v_3(x) = \int_0^x \left(\frac{t^2 - xt}{x}\right)\left[\sum_{n=0}^{2} v_n''(t) + \frac{2}{t}\sum_{n=0}^{2} v_n'(t) + \sum_{n=0}^{2} u_n^2(t) - \sum_{n=0}^{2} v_n^2(t) - 6\sum_{n=0}^{2} v_n(t) - 6 + 6t^2\right]dt = \\ -\frac{1}{2340}x^{12} - \frac{137}{23100}x^{10} - \frac{19}{840}x^8 - \frac{1}{35}x^6 \end{cases} \quad \ldots(50)$$

$\vdots$

The series solution is then obtain by summing the above iterations

$$\begin{cases} u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \ldots = \\ 1 + 2x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{11}{280}x^8 + \frac{137}{23100}x^{10} + \frac{1}{2340}x^{12} + \ldots \\ v(x) = v_0(x) + v_1(x) + v_2(x) + v_3(x) + \ldots = \\ -1 - \frac{1}{2}x^4 - \frac{1}{6}x^6 - \frac{11}{280}x^8 - \frac{137}{23100}x^{10} - \frac{1}{2340}x^{12} + \ldots \end{cases} \quad \ldots(51)$$

**Table 4: Numerical Results for Approximate Solution of $u(x)$ in Example 3**

| x | Exact Solution u(x) | MNIM u(x) | VIM u(x) | NIM u(x) | MNIM Error | VIM Error | NIM Error |
|---|---|---|---|---|---|---|---|
| **0** | 1.00 | 1.00000 | 1.00000 | 1.00000 | 0 | 0 | 0 |
| **0.1** | 1.02 | 1.02005 | 1.02005 | 1.02003 | 2.40505E-11 | 2.88E-08 | 2.003E-05 |
| **0.2** | 1.08 | 1.08081 | 1.08081 | 1.08049 | 6.34524E-09 | 1.89E-06 | 0.0003219 |
| **0.3** | 1.18 | 1.18417 | 1.18415 | 1.18253 | 1.70922E-07 | 2.25E-05 | 0.0016425 |

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **0.4** | 1.33 | 1.33351 | 1.33338 | 1.32826 | 1.82899E-06 | 0.000134 | 0.0052543 |
| **0.5** | 1.53 | 1.53401 | 1.53347 | 1.52097 | 1.18941E-05 | 0.000553 | 0.0130526 |
| **0.6** | 1.79 | 1.79327 | 1.79152 | 1.76560 | 5.67743E-05 | 0.001807 | 0.0277265 |
| **0.7** | 2.12 | 2.12210 | 2.11726 | 2.06924 | 0.000219866 | 0.005059 | 0.0530787 |
| **0.8** | 2.54 | 2.53575 | 2.52380 | 2.44188 | 0.000732988 | 0.012684 | 0.0946038 |
| **0.9** | 3.06 | 3.05572 | 3.02861 | 2.89739 | 0.002184659 | 0.029294 | 0.1605141 |
| **1** | 3.72 | 3.71231 | 3.65476 | 3.45476 | 0.005971361 | 0.06352 | 0.2635199 |

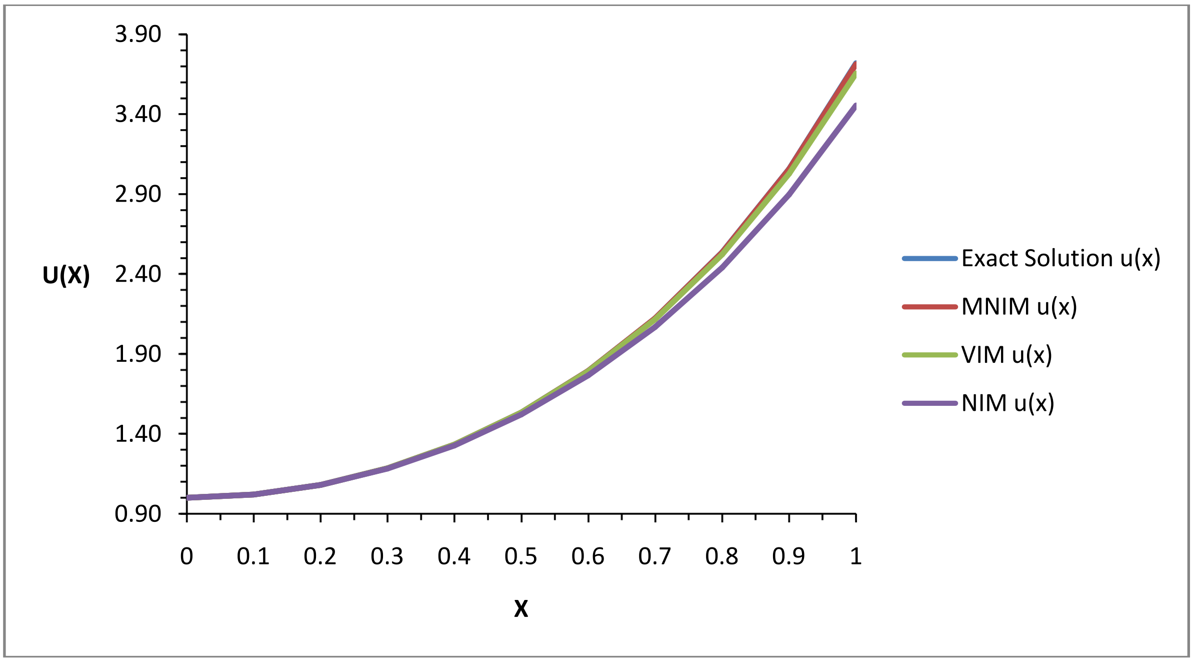


Figure 4: Plots of the exact solution $u(x)$ , NIM, VIM and the MNIM.

**Table 5: Numerical Results for Approximate Solution of $v(x)$ in Example 3**

| x | Exact Solution v(x) | MNIM v(x) | VIM v(x) | NIM v(x) | MNIM Error | VIM Error | NIM Error |
|---|---|---|---|---|---|---|---|
| **0** | -1.00 | -1.00000 | -1.00000 | -1.00000 | 0 | 0 | 0 |
| **0.1** | -1.00 | -1.00005 | -1.00005 | -1.00005 | 2.4051E-11 | 2.464E-11 | 2.464E-11 |
| **0.2** | -1.00 | -1.00081 | -1.00081 | -1.00081 | 6.3452E-09 | 6.953E-09 | 6.954E-09 |
| **0.3** | -1.00 | -1.00417 | -1.00417 | -1.00417 | 1.7092E-07 | 2.059E-07 | 2.062E-07 |
| **0.4** | -1.01 | -1.01351 | -1.01351 | -1.01351 | 1.829E-06 | 2.451E-06 | 2.458E-06 |
| **0.5** | -1.03 | -1.03401 | -1.03401 | -1.03401 | 1.1894E-05 | 1.769E-05 | 1.779E-05 |
| **0.6** | -1.07 | -1.07327 | -1.07324 | -1.07324 | 5.6774E-05 | 9.264E-05 | 9.357E-05 |
| **0.7** | -1.14 | -1.14210 | -1.14193 | -1.14192 | 0.00021987 | 0.0003874 | 0.0003933 |
| **0.8** | -1.26 | -1.25575 | -1.25511 | -1.25508 | 0.00073299 | 0.0013698 | 0.0013992 |
| **0.9** | -1.44 | -1.43572 | -1.43366 | -1.43353 | 0.00218466 | 0.0042526 | 0.0043733 |
| **1** | -1.72 | -1.71231 | -1.70638 | -1.70595 | 0.00597136 | 0.0119021 | 0.0123294 |

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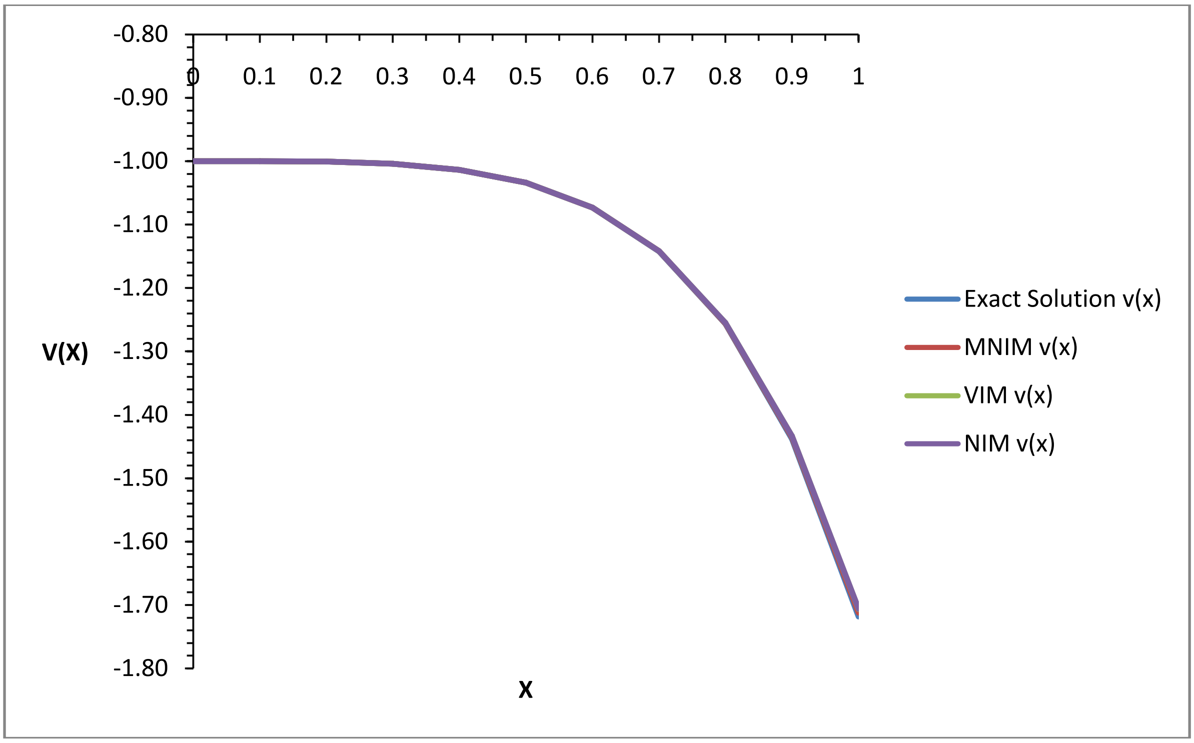


Figure 5: Plots of the exact solution $v(x)$, NIM, VIM and the MNIM

## VII. DISCUSSION OF RESULTS

The main objective of this work has been achieved by solving some linear and nonlinear system of Emden-Fowler Type equation by the proposed method. This purpose has been obtained by implementing the modified NIM. In addition, the comparison between the results obtained by the proposed method, the exact solution and results obtained by other methods such as NIM and VIM was presented as shown in tables and figures. The MNIM is efficient and reliable to find the approximate solution of some linear and nonlinear systems of Emden-Fowler Type equation (see Table 1-5) and exact solution of some systems of Emden-Fowler Type equation (see, example 1). The proposed methods did not require any assumption to deal with the nonlinear terms unlike VIM and the ADM that requires the so-called Adomian polynomial and the Lagrange's Multiplier for the nonlinear case. When comparing the results of the NMIM with those of the NIM and VIM the numerical solutions obtained by MNIM are more accurate (see table and fig) there in. The approximate error will decreases when there are more iteration which are clarified in the figures and tables there in. In comparing the results obtained by the MNIM with those of the existing methods, it is observed in general that the approximate solutions obtained by the MNIM converge faster without any restricted assumptions and possesses high-order accuracy (see Table 1-5, figure 1-5).

## VIII. CONCLUSION

In this work, a semi-analytical method based on the NIM and the VIM was introduced, then it is used to solve nonlinear systems of Emden-Fowler Type equations. To support the analysis, two nonlinear systems of Emden-Fowler Type equations and two linear systems of Emden-Fowler Type equations are solved. The obtained results reveal that this method is simpler and shorter in its computational procedures and time than the other methods. Therefore, this method is more suitable and convenient for solving nonlinear problems.

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