International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 5, pp: 866-879 www.ijaem.net ISSN: 2395-5252

A simple and elegant proof for Fermat's last Theorem

G. Kesavan Nair.

Date of Submission: 30-08-2020 Date of Acceptance: 12-09-2020

Permanent Affiliations:

I worked in the prestigious VIKRAM SARABHAI SPACE CENTRE (VSSC), INDIAN SPACE RESEARCH ORGANOSATION (ISRO), TRIVANDRUM, for 39 years and seven months from 1971 onwards. I was responsible for handling and managing of Main-Frame computers, latest Servers and Workstations and network of the Central Computer Facility in VSSC. I also worked as a designer and programmer in Fortran Language. I designed a set up of Cross-assembler, linkage editor, Simulator, Real-time Executive and Console Service routine all of which were used for the development of an On-Board Computer (bit-Slice) in Main-Frame system in the year 1982-83. The programs for Cross-Assembler and the simulator were written solely by me based on my design.

I was interested in finding a solution for Fermat's Last theorem from 1981 onwards. I was partly successful in 1982 itself. After retirement from office, I could dedicate more time for research. I found two entirely different solutions for the problem – the first one was published in Amazon Kindle direct Publishing (KDP). The second one is a follow up of my solution done in 1982. I am interested in publishing my solution in a Mathematical Journal of International reputation for getting attention from great Mathematicians, as there is no prize money involved.

My publications include:

1. Magic Squares – All combinations of 3rd and 4th order. 2012

Print Length: 274 pages Page Numbers Source ISBN: 1469981890

ASIN: B00AENI8LG, Format: Kindle Edition

2. **SUDOKU - the combinations**

Paperback, 280 pages, Published March 30th 2012 by Createspace Independent Publishing Platform, **ISBN** 1475080999

3. Compact Ad Calendars for 2500 Years January 2012

Paperback: 32 pages; **Publisher:** Createspace Independent Pub

ISBN-10: 1468174886, **ISBN-13:** 978-1468174885

4. EQUATIONS FOR PYTHAGOREAN TRIPLES & Fermat's last Theorem – a new proof, and Andrew Beal's conjecture – proof for special case , Sep 16, 2019

ISBN-10: 1693852969, **ISBN-13:** 978-1693852961

Name of Author: **G. Kesavan Nair.**

Address: **TC 25/2727(5), PRA L 9, LUKE'S LANE,**
PUTHENCHANTHAI, TRIVANDRUM – 695001,
KERALA, INDIA.
Phone: 0471-2470191 Mobile: 9061717072

ABSTRACT

The problem popularly known as Fermat's last Theorem ("Prove $a^n = b^n + c^n$ for n greater than 2 does not exist") has remained unsolved for about 350 years.

I converted the statement of the problem to "Prove $a^n - b^n - c^n = 0$ for **n** greater than 2 does not exist", which means the same. A variable m is introduced which is always even and m = b + c – a. The equality $a^n - b^n - c^n = 0$ is established as a turning point which is the breakthrough for solving the problem.

Key words: Fermat Lost Proof 350 years Old

AMS Subject Classification: 11-XX Number theory

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 5, pp: 866-879 www.ijaem.net ISSN: 2395-5252

I. INTRODUCTION

Pierre Fermat (1601-1655), a busy lawyer, came across a copy of the book "THE ARITHMETIC OF DIOPHANTUS "when he was 30 years old. Fermat developed a hobby of working out the ancient problems of DIOPHANTUS and putting in the margins of the book notes of any proofs he had discovered. In connection with the equation $a^2 = b^2 + c^2$ DIOPHANTUS had a problem - "Divide a square into two squares." Fermat, generalising this wrote in the margin "On the other hand it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates or generally any power except a square into two powers of the same exponent. I have discovered a truly marvellous proof of this, which however the margin is not large enough to contain." This marvellous proof of Fermat was never found.

Mathematicians have been trying ever since Fermat's death to rediscover the proof or prove Fermat wrong. The Gottingen Academy of Sciences had given the **Wolfskehl prize** to **Andrew Wiles** in 1997 for solving the problem.

Here, I give a very simple and elegant proof created by me that supports Fermat's statement.

It has already been established that it is only needed to prove **Fermat's last Theorem** for mutually prime values of a, b, and c. This is because

If $(d \times a)^n = (d \times b)^n + (d \times c)^n$ then $\mathbf{d^n}$ can be taken out as common factor to reach the equality with mutually prime values a, b, and c.

Let us consider only mutually prime values for a, b, and c, all values being different. No two of a, b, and c can be even as they are mutually prime. Only one of them is even and the other two are odd. Let a, and c, be odd and b, even.

Let **a** be the biggest number, **b** smaller than **a**, and **c** smaller than **b**.

The case where **a**, is even, **b**, and **c**, odd will be considered later.

Let us introduce a new variable **m**, which is equal to (b + c) – a. As **a**, is odd and (**b + c**), also is odd, **m** = (**b + c**) – **a**, is always **even**. When **a**, is even, **b** and **c** are odd, then also (**b + c**) – **a**, is **even**.

Depending on the values of a, b, and c, **m can take any even number value from 0 to infinity**. So, **m** can be written as **2n – 2** where **n** takes values from 1 to infinity. As a general even number **2n – 2** is taken to include 0 as the starting value of **m**.

Let us first redefine the problem in order to simplify the solution. The actual problem "Prove $a^n = b^n + c^n$ for n greater than 2 does not exist" can be rewritten as "Prove $a^n - b^n - c^n = 0$ for **n** greater than 2 does not exist". Both mean the same thing.

Turning Point nature of $a^n - b^n - c^n$

For different values of **a, b, and c** it is possible to have $a^n - b^n - c^n > \mathbf{0}$, or $< \mathbf{0}$ and for some n, $a^n - b^n - c^n$ equal to zero. The variable **n** is reused as it also takes values from 1 to infinity.

$a^n - b^n - c^n = 0$ is a **zero-value turning point** for values of **n, a, b, and c**, as for those values of **a, b, and c**, only, the value of $(a^n - b^n - c^n)$ becomes positive for the **very first time** for a particular value of **n**. At the turning point, the value of $a^n - b^n - c^n$ becomes positive for the very first time. For these values of **a, b, and c**, there will also be cases where

$(a - 2)^n - (b - 2)^n - c^n < \mathbf{0}$ and,

$(a + 2)^n - (b + 2)^n - c^n > \mathbf{0}$

Here, the values of **a** and **b** only are altered by 2 to keep the value of **m** = **(b + c) – a** the same and keep **a** as odd and **b** as even to maintain **a, b, and c** mutually prime..

The equality $a^n - b^n - c^n = 0$ if it exists, lies between

$(a - 2)^n - (b - 2)^n - c^n$ and

$(a + 2)^n - (b + 2)^n - c^n$.

Another way to say that $a^n - b^n - c^n = 0$, if it exists, is a turning point is by saying that it lies between $a^{n-1} - b^{n-1} - c^{n-1} < \mathbf{0}$ and $a^{n+1} - b^{n+1} - c^{n+1} > \mathbf{0}$

If $a^n - b^n - c^n = \mathbf{0}$ exists then, for all **k** less than **n**, $(a^k - b^k - c^k)$ should be less than 0 and for all **k** greater than **n,** $(a^k - b^k - c^k)$ should be greater than 0**.** This is true for the **positive-value turning point** also.

Detailed numerical study on turning point nature is given in Appendix 2.

Let us find 3 consecutive values for **a, b, and c** in terms of **m.**

In this case (**a – b**) and (**b – c**) are equal to 1 and the values of **a, b, c** form a smallest possible set for a particular value of **m** as they are close together.

The value of **a** is greater than **b**, and **b** is greater than **c** for the equality $a^n - b^n - c^n = 0$, interchanging **b** and **c** if necessary.

The values of **a, b, and c** for a particular value of **m** can be written as

a = m + 3,

b = m + 2, and

c = m + 1.

Here **m** is equal to **2n – 2**, an even number depending on the value of **n** which can be from 1 to infinity as stated earlier.

a = m + 3

a = 2n – 2 + 3 (as m = 2n – 2)

a = **2n + 1** **Equation {1}**

b = m + 2

b = 2n – 2 + 2 (as m = 2n – 2)

b = **2n** **Equation {2}**

c = m + 1

c = 2n – 2 + 1 (as m = 2n – 2)

c = **2n - 1** **Equation {3}**

From the above equations it is also seen that **b = c + 1**, and **a = b + 1**. The values of **a** and **b** depend on **c**.

m = (b + c) - a

m = {2n + (2n - 1)} - (2n + 1) (as b = 2n, c = 2n – 1, and a = 2n + 1)

m = **2n – 2**

Equation {4}

This value of m is even for all values of n. Also, this becomes the minimum required value for a turning point value of $(a^n - b^n - c^n)$ for any value of n.

The values of **a, b, and c** provide three consecutive integer values.

When **n = 1** the values of **a, b, c** are,

a = 2n + 1 = **3,**

b = 2n = **2**, and

c = 2n – 1 = **1**.

The value of **m** = (**b** + **c**) - **a** = (2 + 1) – 3 **= 0** for this case.

From the **Equation {4}** for **m** also, the value of **m** = 2n - 2 = 2 x 1 – 2 = 2 - 2 **= 0.**

$a^n - b^n - c^n = \mathbf{0}$.

$3^1 - 2^1 - 1^1 = \mathbf{0}$.

The above is a **zero-value turning point for n = 1, a = 3, b = 2, c = 1, and m=0**. And this turning point zero has occurred for the smallest set of integer values for **a, b, and c**.

For these values of **a, b, c**, and **n = 2**, $a^n - b^n - c^n = 3^2 – 2^2 - 1^2 = 9 – 4 - 1 = 4$, a positive value, even for the smallest possible values of a, b, and c. Hence, it is not possible for having a zero-value turning point for n = 2 when m = 0. In fact for m = 0, it is not possible for having a zero value turning point for any n greater than 1. This is because the minimum required value of **m** for a turning point is **2n – 2** which cannot be zero for **n** greater than 1.

It can be stated that for m = 0 and all its multiples (all of which happen to be zero), the zero-value turning point for $(a^n - b^n - c^n)$ **can occur for one and only one value of n, that is for n = 1. This is KESAVAN NAIR's Postulate number one for zero-value turning points.**

The turning point values of **a, b, and c** depend on the value of **n**, based on the equations {1}, {2}, and {3}.and for n greater than 1, no two turning point can be equal.

If we increase **a** and **b** by 2 we get

$5^1 - 4^1 - 1^1 = \mathbf{0}$.

Here also the value of **m** is zero as in the equation for **m = (b + c) - a**. The value of **b** is positive and **a** is negative and the increase gets cancelled. Infinite mutually prime values for **a** and **b** can be got by increasing **a**, and **b**, by 2 repeatedly and keeping **c** fixed in order to keep **m** fixed at zero. Adding an even number keeps **a** as odd and **b** as even.

In the case where **a**, is even, **b** and **c** odd also, it is possible to have a zero-value turning point for **n = 1**. The smallest possible values we can get for **a, b, and c** are **4, 3, and 1**, ie $4^1 - 3^1 - 1^1 = \mathbf{0}$. The only difference is that the numbers 3 and 1 are with a difference of 2. Adding 2 to **a**, and **b**, we get a = 6, b = 5, and c = 1 and the turning point value $6^1 - 5^1 - 1^1 = \mathbf{0}$. In this way also infinite zero-value turning points can be got for n = 1 and m = 0.

If we increase **c** by 2, we get a = 6, b = 5, and c = 3. The Value of **m** becomes 2. The value of $6^2 – 5^2 - 3^2 = 36 – 25 – 9 = 2$, a **positive value-turning point** for the minimum values of **a, b, c** when **a**, is even **b** and **c** odd, and **n = 2**. As the positive turning point value has come for the very first set of **a, b, and c**, no zero-value turning point can occur for **n** greater than 1, **when a is even, b and c odd.**

Coming back to the case where **a** and **c** are odd and **b** even, we have to increase the value of **c** by 2 to get the next value of **m**, as adding 2 to a and b creates no change in **m**. After adding 2, the value of **c** becomes 3, value of **b**, becomes **c + 1** = 4, value of **a**, becomes **b + 1** = 5, the value of **m** becomes = (**b + c**) – **a** = (4 + 3) – 5 = **2.**

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 5, pp: 866-879 www.ijaem.net ISSN: 2395-5252

As **m** = 2, **2n - 2** = 2, **2n** = 4 or **n = 2**.
The value of $a^2 - b^2 - c^2$ **becomes** $5^2 - 4^2 - 3^2 = 0$**, a zero-value turning point for n = 2. The zero-value turning point exist for all non-zero multiples of m, only the values of a, b, and c will not be mutually prime in those cases.**

Equation for turning point for all n : The general case

Let us consider the **general case** for the turning point ($a^n - b^n - c^n$) for the smallest values of **a, b, c and m**. We have **a = 2n + 1, b = 2n,** and **c = 2n -1** from **Equation {1}, Equation {2}, and Equation {3}.** Let us find $a^n - b^n - c^n$ algebraically.

$$a^n = (2n + 1)^n$$
$$= (2n)^n + nC_1\,(2n)^{(n-1)}\,1^1 + nC_2\,(2n)^{(n-2)}\,1^2 + nC_3\,(2n)^{(n-3)}\,1^3 + \ldots$$
$$b^n = (2n)^n$$
$$c^n = (2n - 1)^n$$
$$= (2n)^n - nC_1\,(2n)^{(n-1)}\,1^1 + nC_2\,(2n)^{(n-2)}\,1^2 - nC_3\,(2n)^{(n-3)}\,1^3 + \ldots$$
$$\mathbf{a^n - b^n - c^n}$$
$$= -(2n)^n + 2nC_1\,(2n)^{(n-1)}\,1^1 + 2nC_3\,(2n)^{(n-3)}\,1^3 + \ldots$$
$$= -(2n)^n + (2n) \times (2n)^{(n-1)} + 2nC_3\,(2n)^{(n-3)}\,1^3 + \ldots$$
$$\mathbf{(a^n - b^n - c^n) = 2nC_3\,(2n)^{(n-3)} + 2nC_5\,(2n)^{(n-5)} + \ldots} \qquad \textbf{Equation \{5\}}$$

The right hand side of the **equation {5}** vanishes when **n** = 2,
Showing that $a^2 - b^2 - c^2 = 0$ or $\mathbf{a^2 = b^2 + c^2}$ **is possible.**

When **n**, is greater than 2 the right hand side of the Equation {5} is **always a positive value** and cannot be zero. So the very first turning point is always a positive value and not zero for all values of **n** greater than 2, the value of **m**, being equal to (**2n – 2**), which depends on **n**.

To get the prior value of **a**, **b** and **c** for the same value of **m**, the number 2 can be subtracted from a, and b, without changing c.

a = (2n + 1) - 2 = **(2n - 1)**
b = **2n - 2** and
c = **(2n - 1)**

$$a^n - b^n - c^n = (2n - 1)^n - (2n - 2)^n - (2n - 1)^n$$
$$\mathbf{a^n - b^n - c^n = -(2n - 2)^n} \qquad \textbf{Equation \{6\}}$$

This is always negative for all values of **n** showing that,

a = (2n + 1),
b = 2n, and
c = (2n - 1)

Are the minimum values for the turning point $a^n - b^n - c^n$ **= 0.**

The Equation {5} and Equation {6} establish the fact that the values of **a, b, and c** got in Equation {1}, Equation {2}, and Equation {3}, are the **very first turning point values** of **a**, **b**, and **c** for ($a^n - b^n - c^n$) for all **n**. Only in the case of n = 1 and n = 2 the turning point value is zero. For all other values of **n** greater than 2, a^n, is greater than ($b^n + c^n$) and ($a^n - b^n - c^n$) is a positive value at the turning point. This is valid for the infinite values of **n**.

Once a^n, has become greater than ($b^n + c^n$) at the very first turning point, any increase in the values of **a and b** by multiples of 2 will cause the difference ($a^n - b^n - c^n$) to become more and more. As **c**, is not altered, the value of **m** = **(b + c) - a**, remains the same.

The right-hand side of the Equation {5} contains only positive terms for **n** greater than 2. Hence it will always give different value for different **n**, increasing with the value of **n**. So, no two turning points for ($a^n - b^n - c^n$) can be equal for different values of n.

The equation {5} is valid for **a, b, and c** that are minimum values obtainable for each n. There are infinite turning points for mutually prime values of **a, b, and c**.

For every odd value of **c** greater than or equal to **2n – 1**, turning points exist. It is possible to have zero-value turning points for mutually prime values of **a, b, c, and n = 2,** for all non-zero multiples of **m** as shown in the summary of result obtained from Appendix 1. For **n** greater than 2, there are no zero-value turning points as shown in the worked out examples in Appendix 1.

For all n greater than 2, the turning point values are different and greater than zero.

The details of calculations are in Appendix 1.

Turning point value for m greater than the minimum value (2n - 2)

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 5, pp: 866-879 www.ijaem.net ISSN: 2395-5252

When we add two to the minimum value of m = 2n - 2, we get m = 2n. Let us have new and general values for a, b, and c. In the general values a, b and c need not be close together as in the case where minimum values only were considered. Here also a is greater than b, and b is greater than c. The values of a and c are odd, and b even, let us have three variables i, j, and k where i is greater than j which is greater than k, which maintain the same hierarchy as a, b, and c.

Let us write the general values of a, b, and c in terms of m, maintaining a, b, and c, mutually prime.

As a is odd a = m + an odd number = 2n + (2i + 1) = ((2n + 2i) – 1)

As b is even b = m + an even number = (2n + 2j), and = (2n + 2j)

As c is odd c = m + an odd number = 2n + (2k + 1) = ((2n + 2k) + 1)

$\mathbf{a^n = ((2n + 2i) + 1)^n}$

$= (2n + 2i)^n + nC_1\,(2n + 2i)^{(n-1)} \times 1^1 + nC_2\,(2n + 2i)^{(n-2)} \times 1^2 + nC_3\,(2n + 2i)^{(n-3)} \times 1^3 + \ldots$

$\mathbf{b^n = (2n + 2j)^n}$

$\mathbf{c^n = ((2n + 2k) + 1)^n}$

$= (2n + 2k)^n + nC_1\,(2n + 2k)^{(n-1)} \times 1^1 + nC_2\,(2n + 2k)^{(n-2)} \times 1^2 + nC_3\,(2n + 2k)^{(n-3)} \times 1^3 + \ldots$

$a^n – b^n – c^n =$

$\mathbf{= [(2n + 2i)^n - (2n + 2j)^n] + nC_1\,[(2n + 2i)^{(n-1)} - (2n + 2k)^{(n-1)}] + nC_2\,[(2n + 2i)^{(n-2)} - (2n + 2k)^{(n-2)}] + nC_3\,[(2n + 2i)^{(n-3)} - (2n + 2k)^{(n-3)}] + \ldots}$ **Equation {7}**

The terms on the right hand side are grouped two at a time, the first one positive and the second is negative. As the variable i is greater than the variable j and the variable k, all the grouped terms are positive for all n greater than 2. Such terms are added together to get the value of $a^n – b^n – c^n$. So it can never be zero, and the value of $a^n – b^n – c^n$ increases with the value of n. The zero-value turning point has already occurred for n = 2 and m = 2, for no other value of n and m greater than 2 can have a zero-value turning point for $a^n – b^n – c^n$.

When there is no zero-value turning point for $a^n – b^n – c^n$, when n is greater than 2, $a^n = b^n + c^n$ also cannot be true for n greater than 2 – proving Fermat's last theorem.

It can be stated that for m = 2 and all its non-zero multiples, the zero-value turning point for ($a^n - b^n - c^n$**) can occur for one and only one value of n, that is for n = 2 for the minimum set of values of a, b, and c which depend on n. This is KESAVAN NAIR's Postulate number two for zero-value turning points.**

II. CONCLUSION

As the right hand side of the **Equation {5} and Equation {7} has only positive terms, when** n increases, the turning point value ($\mathbf{a^n - b^n – c^n}$) also increases as the number of terms increase, making no two turning point values equal when **n** is greater than 2. This is true not only for the very first turning point but also for all other turning points obtained in the worked out examples.

This means that $\mathbf{a^n - b^n – c^n}$ = 0 does not exist for **n** greater than 2. In other words, the equality $\mathbf{a^n = b^n + c^n}$ does not exist for **n** greater than 2 proving Fermat's theorem.

III. APPENDIX 1

Worked out examples

For brevity, the cases considered below are for values of **n** from 2 to 5, and values of **c** from 3 to 15. The method to get the turning point values for **a, b, and c** is given below. 5, 4, 3 is the very first zero-value turning point for **n = 2, m = 2** and the number of steps needed to reach it is **k = 1.**

n	m	k	a	b	c	a^2	b^2	c^2	$b^2 + c^2$	$a^2 - b^2 - c^2$
2	2	1	5	4	3	25	16	9	25	0

Increasing **c** by 2, also increases **a** by 2, **b** by 2, as **b = c + 1, a = b + 1**, keeping **a, b, and c** mutually prime and different, and the difference **m = (b + c) - a** increasing by 2. The value of **m** becomes 4. The starting values for **b** and **a**, are **b = c + 1** and a = **b + 1**. For the 1st step, the value of $\mathbf{a^2 - b^2 - c^2}$ is $\mathbf{7^2 - 6^2 - 5^2}$ = -12 and not a **turning point as turning point is the value for which the equation turns positive for the very first time.** The next value of **a and b**, are got by adding two to **a** and **b** and keeping **c** without change to keep **m** = (b + c) – a same at 4. The value of $\mathbf{a^2 - b^2 - c^2}$ is negative. The step of adding 2 to **a**, and **b**, is repeated. At step 4, an equality turning point value is got for **a = 13, b = 12, and c = 5.**

n	m	k	a	b	c	a^2	b^2	c^2	$b^2 + c^2$	$a^2 - b^2 - c^2$
2	4	1	7	6	5	49	36	25	61	-12

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 5, pp: 866-879 www.ijaem.net ISSN: 2395-5252

2	4	2	9	8	5	81	64	25	89	-8
2	4	3	11	10	5	121	100	25	125	-4
2	4	**4**	**13**	**12**	**5**	**169**	**144**	**25**	**169**	**0**

Any further increase in **a**, and **b**, will make the difference more positive as the difference **steadily increases with every step.**

To get another turning point, **c** has to be increased by 2 to 7. The value of **b** and **a**, becomes 8 and 9 as **b** = **c + 1, a = b + 1**. The value of **m** = (**b** + **c**) – **a**, becomes 6, the next even number. The value for $\mathbf{a^2 - b^2 - c^2}$ becomes $\mathbf{9^2 - 8^2 - 7^2}$ = -32. Repeating the increase of a and b by two at ninth step an equality turning point is got at **a = 25, b = 24, and c = 7.**

n	**m**	**k**	**a**	**b**	**c**	$\mathbf{a^2}$	$\mathbf{b^2}$	$\mathbf{c^2}$	$\mathbf{b^2 + c^2}$	$\mathbf{a^2 - b^2 - c^2}$
2	6	1	9	8	7	81	64	49	113	-32
2	6	2	11	10	7	121	100	49	149	-28
2	6	3	13	12	7	169	144	49	193	-24
2	6	4	15	14	7	225	196	49	245	-20
2	6	5	17	16	7	289	256	49	305	-16
2	6	6	19	18	7	361	324	49	373	-12
2	6	7	21	20	7	441	400	49	449	-8
2	6	8	23	22	7	529	484	49	533	-4
2	6	**9**	**25**	**24**	**7**	**625**	**576**	**49**	**625**	**0**

Here is one more example to clarify the steps required to find the zero-value turning points. Increase c by 2 to make it 9. The initial and smallest value of b and a become 10 and 11. The turning point value got is negative, ie. -60. Repeating the steps of increasing a and b by 2, at step number 16 an equality turning is got for a = 41, b = 40, and c = 9.

n	**m**	**k**	**a**	**b**	**c**	$\mathbf{a^2}$	$\mathbf{b^2}$	$\mathbf{c^2}$	$\mathbf{b^2 + c^2}$	$\mathbf{a^2 - b^2 - c^2}$
2	8	1	11	10	9	121	100	81	181	-60
2	8	2	13	12	9	169	144	81	225	-56
2	8	3	15	14	9	225	196	81	277	-52
2	8	4	17	16	9	289	256	81	337	-48
2	8	5	19	18	9	361	324	81	405	-44
2	8	6	21	20	9	441	400	81	481	-40
2	8	7	23	22	9	529	484	81	565	-36
2	8	8	25	24	9	625	576	81	657	-32
2	8	9	27	26	9	729	676	81	757	-28
2	8	10	29	28	9	841	784	81	865	-24
2	8	11	31	30	9	961	900	81	981	-20
2	8	12	33	32	9	1089	1024	81	1105	-16
2	8	13	35	34	9	1225	1156	81	1237	-12
2	8	14	37	36	9	1369	1296	81	1377	-8
2	8	15	39	38	9	1521	1444	81	1525	-4
2	**8**	**16**	**41**	**40**	**9**	**1681**	**1600**	**81**	**1681**	**0**

Another interesting point to note in the case of n = 2 is that the number of steps required to get the next turning point is predictable. For example, the next turning point value will take 25 steps. The equation is **c + k** which is equal to 16 + 9 = 25.

n	**m**	**k**	**a**	**b**	**c**	$\mathbf{a^2}$	$\mathbf{b^2}$	$\mathbf{c^2}$	$\mathbf{b^2 + c^2}$	$\mathbf{a^2 - b^2 - c^2}$
2	10	1	13	12	11	169	144	121	265	-96
2	10	2	15	14	11	225	196	121	317	-92
2	10	3	17	16	11	289	256	121	377	-88
2	10	4	19	18	11	361	324	121	445	-84
2	10	5	21	20	11	441	400	121	521	-80

2	10	6	23	22	11	529	484	121	605	-76
2	10	7	25	24	11	625	576	121	697	-72
2	10	8	27	26	11	729	676	121	797	-68
2	10	9	29	28	11	841	784	121	905	-64
2	10	10	31	30	11	961	900	121	1021	-60
2	10	11	33	32	11	1089	1024	121	1145	-56
2	10	12	35	34	11	1225	1156	121	1277	-52
2	10	13	37	36	11	1369	1296	121	1417	-48
2	10	14	39	38	11	1521	1444	121	1565	-44
2	10	15	41	40	11	1681	1600	121	1721	-40
2	10	16	43	42	11	1849	1764	121	1885	-36
2	10	17	45	44	11	2025	1936	121	2057	-32
2	10	18	47	46	11	2209	2116	121	2237	-28
2	10	19	49	48	11	2401	2304	121	2425	-24
2	10	20	51	50	11	2601	2500	121	2621	-20
2	10	21	53	52	11	2809	2704	121	2825	-16
2	10	22	55	54	11	3025	2916	121	3037	-12
2	10	23	57	56	11	3249	3136	121	3257	-8
2	10	24	59	58	11	3481	3364	121	3485	-4
2	**10**	**25**	**61**	**60**	**11**	**3721**	**3600**	**121**	**3721**	**0**

The next equality turning point will be got at step 25 + 11 = 36.

n	**m**	**k**	**a**	**b**	**c**	a^2	b^2	c^2	$b^2 + c^2$	$a^2 - b^2 - c^2$
2	12	1	15	14	13	225	196	169	365	-140
2	12	2	17	16	13	289	256	169	425	-136
2	12	3	19	18	13	361	324	169	493	-132
2	12	4	21	20	13	441	400	169	569	-128
2	12	5	23	22	13	529	484	169	653	-124
2	12	6	25	24	13	625	576	169	745	-120
2	12	7	27	26	13	729	676	169	845	-116
2	12	8	29	28	13	841	784	169	953	-112
2	12	9	31	30	13	961	900	169	1069	-108
2	12	10	33	32	13	1089	1024	169	1193	-104
2	12	11	35	34	13	1225	1156	169	1325	-100
2	12	12	37	36	13	1369	1296	169	1465	-96
2	12	13	39	38	13	1521	1444	169	1613	-92
2	12	14	41	40	13	1681	1600	169	1769	-88
2	12	15	43	42	13	1849	1764	169	1933	-84
2	12	16	45	44	13	2025	1936	169	2105	-80
2	12	17	47	46	13	2209	2116	169	2285	-76
2	12	18	49	48	13	2401	2304	169	2473	-72
2	12	19	51	50	13	2601	2500	169	2669	-68
2	12	20	53	52	13	2809	2704	169	2873	-64
2	12	21	55	54	13	3025	2916	169	3085	-60
2	12	22	57	56	13	3249	3136	169	3305	-56
2	12	23	59	58	13	3481	3364	169	3533	-52
2	12	24	61	60	13	3721	3600	169	3769	-48
2	12	25	63	62	13	3969	3844	169	4013	-44
2	12	26	65	64	13	4225	4096	169	4265	-40
2	12	27	67	66	13	4489	4356	169	4525	-36
2	12	28	69	68	13	4761	4624	169	4793	-32
2	12	29	71	70	13	5041	4900	169	5069	-28
2	12	30	73	72	13	5329	5184	169	5353	-24
2	12	31	75	74	13	5625	5476	169	5645	-20

2	12	32	77	76	13	5929	5776	169	5945	-16
2	12	33	79	78	13	6241	6084	169	6253	-12
2	12	34	81	80	13	6561	6400	169	6569	-8
2	12	35	83	82	13	6889	6724	169	6893	-4
2	**12**	**36**	**85**	**84**	**13**	**7225**	**7056**	**169**	**7225**	**0**

One subsequent turning point is when k is equal to 36 + 13 = 49.

n	**m**	**k**	**a**	**b**	**c**	**a^2**	**b^2**	**c^2**	**$b^2 + c^2$**	**$a^2 - b^2 - c^2$**
2	**14**	**49**	**113**	**112**	**15**	**12769**	**12544**	**225**	**12769**	**0**

Here, we are ending the calculation for **n** = 2 with value of **c** = 15.

Using the above method for obtaining zero-value turning point values, it can be shown that for all odd values of c from 3 to infinity, there exists a zero-value turning point for $a^2 - b^2 - c^2$ for mutually prime values of **a, b, and c**.

The mutually prime turning point values of **a, b, and c** obtained for **m** = 2 together with the ones got for all non-zero multiples of m, cover all the possible zero-value turning points for **a, b, c**, and **n** = 2.

Case when n = 3

Now let us consider the case when **n = 3**. Let us start with the smallest value of **m = 2n - 2 = 4**. Now, c = 5, b = c + 1 =6, a = b + 1 = 7. Here, we get the first positive-value turning point as 2 for n = 3 and m = 4. For the previous values of a = 5, b = 4, and c = 5, (**$a^3 - b^3 - c^3$**) is negative. The number of steps k = 1.

n	**m**	**k**	**a**	**b**	**c**	**a^3**	**b^3**	**c^3**	**$b^3 + c^3$**	**$a^3 - b^3 - c^3$**
3	**4**	**1**	**7**	**6**	**5**	**343**	**216**	**125**	**341**	**2**

To get a subsequent turning point, the value of **c** has to be increased by 2 thereby increasing **c** to 7, **b** to 8 and a, to 9, and **m to 6**. The turning point (**$a^3 - b^3 - c^3$**) = 126 is got at step 3 with values of a, b, and c as 13, 12 and 7. The value of m = 2n – 2 = 6.

n	**m**	**k**	**a**	**b**	**c**	**a^3**	**b^3**	**c^3**	**$b^3 + c^3$**	**$a^3 - b^3 - c^3$**
3	6	1	9	8	7	729	512	343	855	-126
3	6	2	11	10	7	1331	1000	343	1343	-12
3	**6**	**3**	**13**	**12**	**7**	**2197**	**1728**	**343**	**2071**	**126**

Repeating the step of incrementing **c** by 2 yields **a** = 11, **b** = 10, **c** = 9, and **m = (b + c) – a = 8**. The turning point value for (**$a^3 - b^3 - c^3$**) is got as 88 at step 4 with values for **a, b, c as 17, 16, and 9** respectively.

n	**m**	**k**	**a**	**b**	**c**	**a^3**	**b^3**	**c^3**	**$b^3 + c^3$**	**$a^3 - b^3 - c^3$**
3	8	1	11	10	9	1331	1000	729	1729	-398
3	8	2	13	12	9	2197	1728	729	2457	-260
3	8	3	15	14	9	3375	2744	729	3473	-98
3	**8**	**4**	**17**	**16**	**9**	**4913**	**4096**	**729**	**4825**	**88**

Repeating the step of incrementing **c** by 2 yields a = 13, b = 12, c = 11, and m = (b + c) – a = 8. The turning point value for (**$a^3 - b^3 - c^3$**) is got as 188 at step 6 with values for **a, b, c as 23, 22, and 11** respectively.

n	**m**	**k**	**a**	**b**	**c**	**a^3**	**b^3**	**c^3**	**$b^3 + c^3$**	**$a^3 - b^3 - c^3$**
3	10	1	13	12	11	2197	1728	1331	3059	-862
3	10	2	15	14	11	3375	2744	1331	4075	-700
3	10	3	17	16	11	4913	4096	1331	5427	-514
3	10	4	19	18	11	6859	5832	1331	7163	-304
3	10	5	21	20	11	9261	8000	1331	9331	-70
3	**10**	**6**	**23**	**22**	**11**	**12167**	**10648**	**1331**	**11979**	**188**

Repeating the step of incrementing c by 2 yields a = 15, b = 14, c = 13, and m = (b + c) – a = 12. The turning point value for ($a^3 - b^3 - c^3$) is go as 240 at step 8 with values for **a, b, c as 29, 28, and 13** respectively.

n	m	k	a	b	c	a^3	b^3	c^3	$b^3 + c^3$	$a^3 - b^3 - c^3$
3	12	1	15	14	13	3375	2744	2197	4941	-1566
3	12	2	17	16	13	4913	4096	2197	6293	-1380
3	12	3	19	18	13	6859	5832	2197	8029	-1170
3	12	4	21	20	13	9261	8000	2197	10197	-936
3	12	5	23	22	13	12167	10648	2197	12845	-678
3	12	6	25	24	13	15625	13824	2197	16021	-396
3	12	7	27	26	13	19683	17576	2197	19773	-90
3	**12**	**8**	**29**	**28**	**13**	**24389**	**21952**	**2197**	**24149**	**240**

Repeating the step of incrementing **c** by 2 yields **a = 17, b = 16, c = 15**, and **m = (b + c) – a = 14.** The turning point value for ($a^3 - b^3 - c^3$) is got as 196 at step 10 with values for a, b, c as 35, 34, and 15 respectively.

n	m	k	a	b	c	a^3	b^3	c^3	$b^3 + c^3$	$a^3 - b^3 - c^3$
3	14	1	17	16	15	4913	4096	3375	7471	-2558
3	14	2	19	18	15	6859	5832	3375	9207	-2348
3	14	3	21	20	15	9261	8000	3375	11375	-2114
3	14	4	23	22	15	12167	10648	3375	14023	-1856
3	14	5	25	24	15	15625	13824	3375	17199	-1574
3	14	6	27	26	15	19683	17576	3375	20951	-1268
3	14	7	29	28	15	24389	21952	3375	25327	-938
3	14	8	31	30	15	29791	27000	3375	30375	-584
3	14	9	33	32	15	35937	32768	3375	36143	-206
3	**14**	**10**	**35**	**34**	**15**	**42875**	**39304**	**3375**	**42679**	**196**

Case when n = 4

For n = 4, we can carry out the above steps to get turning point values for a, b, and c. Only when the value of **m** = 2n – 2 = 6. The value of **c = 7, b = 8, and a = 9.** The very first turning point is got.

n	m	k	a	b	c	a^4	b^4	c^4	$b^4 + c^4$	$a^4 - b^4 - c^4$
4	**6**	**1**	**9**	**8**	**7**	**6561**	**4096**	**2401**	**6497**	**64**

Increasing c to 9, leads to a turning point value at step 2 for n = 4.

n	m	k	a	b	c	a^4	b^4	c^4	$b^4 + c^4$	$a^4 - b^4 - c^4$
4	8	1	11	10	9	14641	10000	6561	16561	-1920
4	**8**	**2**	**13**	**12**	**9**	**28561**	**20736**	**6561**	**27297**	**1264**

Increasing c to 11, leads to a turning point value at step 3 for n = 4.

n	m	k	a	b	c	a^4	b^4	c^4	$b^4 + c^4$	$a^4 - b^4 - c^4$
4	10	1	13	12	11	28561	20736	14641	35377	-6816
4	10	2	15	14	11	50625	38416	14641	53057	-2432
4	**10**	**3**	**17**	**16**	**11**	**83521**	**65536**	**14641**	**80177**	**3344**

Increasing c to 13, leads to a turning point value at step 4 for n = 4.

n	m	k	a	b	c	a^4	b^4	c^4	$b^4 + c^4$	$a^4 - b^4 - c^4$
4	12	1	15	14	13	50625	38416	28561	66977	-16352
4	12	2	17	16	13	83521	65536	28561	94097	-10576
4	12	3	19	18	13	130321	104976	28561	133537	-3216
4	12	4	21	20	13	194481	160000	28561	188561	5920

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 5, pp: 866-879 www.ijaem.net ISSN: 2395-5252

Increasing c to 15, leads to a turning point value at step 5 for n = 4. So, in the case when n = 4, the number of steps required for the next turning point is predictable.

n	m	k	a	b	c	a^4	b^4	c^4	$b^4 + c^4$	$a^4 - b^4 - c^4$
4	14	1	17	16	15	83521	65536	50625	116161	-32640
4	14	2	19	18	15	130321	104976	50625	155601	-25280
4	14	3	21	20	15	194481	160000	50625	210625	-16144
4	14	4	23	22	15	279841	234256	50625	284881	-5040
4	**14**	**5**	**25**	**24**	**15**	**390625**	**331776**	**50625**	**382401**	**8224**

Case when n = 5

For n = 5, we can carry out the above steps to get turning point values for a, b, and c. The value of **m = 2n – 2** = 8. The very first proper turning point is got when m = 8 for n = 5, as that is the minimum required value for m for n=5. The value of c = 9, b = 10, and a = 11.

n	m	k	a	b	c	a^5	b^5	c^5	$b^5 + c^5$	$a^5 - b^5 - c^5$
5	**8**	**1**	**11**	**10**	**9**	**161051**	**100000**	**59049**	**159049**	**2002**

Increasing **c** to 11 finds a turning point at step 2 for **n** = 5.

n	m	k	a	b	c	a^5	b^5	c^5	$b^5 + c^5$	$a^5 - b^5 - c^5$
5	10	1	13	12	11	371293	248832	161051	409883	-38590
5	**10**	**2**	**15**	**14**	**11**	**759375**	**537824**	**161051**	**698875**	**60500**

Increasing **c** to 13 finds a turning point at step 3 for **n** = 5 and **m** = 12.

n	m	k	a	b	c	a^5	b^5	c^5	$b^5 + c^5$	$a^5 - b^5 - c^5$
5	12	1	15	14	13	759375	537824	371293	909117	-149742
5	12	2	17	16	13	1419857	1048576	371293	1419869	-12
5	**12**	**3**	**19**	**18**	**13**	**2476099**	**1889568**	**371293**	**2260861**	**215238**

Increasing **c** to 15 finds a turning point at step 3 for n = 5 and m = 14. No pattern is recognised here. Maybe, we can find some pattern if we consider more cases.

n	m	k	a	b	c	a^5	b^5	c^5	$b^5 + c^5$	$a^5 - b^5 - c^5$
5	14	1	17	16	15	1419857	1048576	759375	1807951	-388094
5	14	2	19	18	15	2476099	1889568	759375	2648943	-172844
5	**14**	**3**	**21**	**20**	**15**	**4084101**	**3200000**	**759375**	**3959375**	**124726**

For other values of n > 5 also, we can find points following the above method. The turning point value steadily increases for every increase in the value of n, thereby making it impossible for $\mathbf{a^n - b^n - c^n}$ = 0 for **n** greater than 2.

Summary of results

For **n** = 2, the value of **m = 2n – 2 = 2**. The first turning point values for a, b, and are 5, 4, and 3. When **c** is incremented by 2, a new set of values for a, b, and c are obtained. Incrementing of c is carried out up to c = 15 and results shown below.

n	m	k	a	b	c	a^2	b^2	c^2	$b^2 + c^2$	$a^2 - b^2 - c^2$
2	2	1	5	4	3	25	16	9	25	0
2	4	2	13	12	5	169	144	25	169	0
2	6	3	25	24	7	625	576	49	625	0
2	8	4	41	40	9	1681	1600	81	1681	0
2	10	5	61	60	11	3721	3600	121	3721	0
2	12	6	85	84	13	7225	7056	169	7225	0
2	14	7	113	112	15	12769	12544	225	12769	0

For n = 3, the value of **m = 2n – 2 = 4**. The first turning point values for **a, b**, and **c** are 7, 6, and 5. The positive-turning point value is 2. When **c** is incremented by 2, a new set of values for **a, b, and c** are obtained. A turning

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 5, pp: 866-879 www.ijaem.net ISSN: 2395-5252

point is got at the 3rd step of incrementing **a and b**. Incrementing of **c** is carried out up to c = 15 and results shown below.

n	m	k	a	b	c	a^3	b^3	c^3	$b^3 + c^3$	$a^3 - b^3 - c^3$
3	4	1	7	6	5	343	216	125	341	2
3	6	2	13	12	7	2197	1728	343	2071	126
3	8	3	17	16	9	4913	4096	729	4825	88
3	10	4	23	22	11	12167	10648	1331	11979	188
3	12	5	29	28	13	24389	21952	2197	24149	240
3	14	6	35	34	15	42875	39304	3375	42679	196

For n = 4, the value of **m = 2n – 2 = 6**. The first turning point values for **a, b**, and **c** are 9, 8, and 7. The positive-turning point value is 64. When **c** is incremented by 2, a new set of values for **a, b, and c** are obtained. Incrementing of **c** is carried out up to c = 15 and results shown below.

n	m	k	a	b	c	a^4	b^4	c^4	$b^4 + c^4$	$a^4 – b^4 – c^4$
4	6	1	9	8	7	6561	4096	2401	6497	64
4	8	2	13	12	9	28561	20736	6561	27297	1264
4	10	3	17	16	11	83521	65536	14641	80177	3344
4	12	4	21	20	13	194481	160000	28561	188561	5920
4	14	5	25	24	15	390625	331776	50625	382401	8224

For n = 5, the value of **m = 2n – 2 = 8**. The first turning point values for **a, b**, and **c** are 11, 10, and 9. The positive-turning point value is 2002. When **c** is incremented by 2, a new set of values for **a, b, and c** are obtained. Incrementing of **c** is carried out up to c = 15 and results shown below.

n	m	k	a	b	c	a^5	b^5	c^5	$b^5 + c^5$	$a^5 - b^5 - c^5$
5	8	1	11	10	9	161051	100000	59049	159049	2002
5	10	2	15	14	11	759375	537824	161051	698875	60500
5	12	3	19	18	13	2476099	1889568	371293	2260861	215238
5	14	4	21	20	15	4084101	3200000	759375	3959375	124726

For all n greater than 2, the turning point values are different and greater than zero.

Appendix 2

The turning point nature of $a^n - b^n – c^n$

There are two methods to demonstrate the turning point nature of $a^n - b^n – c^n$. In the first method a, and b are altered by two, keeping c constant so that the difference b + c – a remains constant as well as keep a, b, and c mutually prime. Here it is found that $(a - 2)^n - (b - 2)^n – c^n$ is negative and $(a + 2)^n - (b + 2)^n – c^n$ is positive. The value of $\mathbf{a^n - b^n – c^n}$ lies between $(a - 2)^n - (b - 2)^n – c^n$ and $(a + 2)^n - (b + 2)^n – c^n$ for the turning point values of a, b, and c. At the turning point values of a, b, and c, the value $\mathbf{a^n - b^n – c^n}$ becomes positive, (or zero in the case of n = 2), for the very first time.

Method1 – numerical examples 1

Method1: Case 1 n = 2 a = 5 b = 4 c = 3

(a - 2) = 3; (b - 2) = 2; c = 3; a = 5; b = 4; c = 3; (a + 2) = 7; (b + 2) = 6; c = 3;

$(a - 2)^2 - (b - 2)^2 - c^2 = 3^2 – 2^2 - 3^2 = 9 – 4 – 9 = -4;$

$\mathbf{a^2 - b^2 - c^2 = 5^2 – 4^2 – 3^2 = 25 – 16 - 9 = 0;}$

$(a + 2)^2 - (b + 2)^2 - c^2 = 7^2 – 6^2 – 3^2 = 49 – 36 - 9 = 4;$

Method1: Case 2 n = 2 a = 13 b = 12 c = 5

(a - 2) = 11; (b - 2) = 10; c = 5; a = 13; b = 12; c = 5; (a + 2) = 15; (b + 2) = 14; c = 5;

$(a - 2)^2 - (b - 2)^2 - c^2 = 11^2 – 10^2 - 5^2 = 121 – 100 – 25 = -4;$

$\mathbf{a^2 - b^2 - c^2 = 13^2 – 12^2 – 5^2 = 169 – 144 - 25 = 0;}$

$(a + 2)^2 - (b + 2)^2 - c^2 = 15^2 – 14^2 – 5^2 = 225 – 196 - 25 = 4;$

Method1: Case 3 n = 2 a = 25 b = 24 c = 7

(a - 2) = 23; (b - 2) = 22; c = 7; a = 25; b = 24; c = 7; (a + 2) = 27; (b + 2) = 26; c = 7;

$(a - 2)^2 - (b - 2)^2 - c^2 = 23^2 – 22^2 - 7^2 = 529 – 484 – 49 = -4$
$\mathbf{a^2 - b^2 - c^2 = 25^2 – 24^2 – 7^2 = 625 – 576 - 49 = 0;}$
$(a + 2)^2 - (b + 2)^2 - c^2 = 27^2 – 26^2 – 7^2 = 729 – 676 - 49 = 4;$

Method1: Case 4 n = 2 a = 41 b = 40 c = 9

(a - 2) = 39; (b - 2) = 38; c = 9; a = 41; b = 40; c = 9; (a + 2) = 43; (b + 2) = 42; c = 9;
$(a - 2)^2 - (b - 2)^2 - c^2 = 39^2 – 38^2 - 9^2 = 1521 – 1444 – 81 = -4$
$\mathbf{a^2 - b^2 - c^2 = 41^2 – 40^2 – 9^2 = 1681 – 1600 - 81 = 0;}$
$(a + 2)^2 - (b + 2)^2 - c^2 = 43^2 – 42^2 – 9^2 = 1849 – 1764 - 81 = 4;$

Method1: Case 5 n = 3 a = 7 b = 6 c = 5

(a - 2) = 5; (b - 2) = 4; c = 5; a = 7; b = 6; c = 5; (a + 2) = 9; (b + 2) = 8; c = 5;

$(a - 2)^3 - (b - 2)^3 - c^3 = 5^3 – 4^3 - 5^3 = 125 – 64 – 125 = -64;$
$\mathbf{a^3 - b^3 - c^3 = 7^3 – 6^3 – 5^3 = 343 – 216 - 125 = 2;}$
$(a + 2)^3 - (b + 2)^3 - c^3 = 9^3 – 8^3 – 5^3 = 729 – 512 - 125 = 92;$

Method1: Case 6 n = 3 a = 13 b= 12 c = 7

(a - 2) = 11; (b - 2) = 10 c = 7; a = 13; b = 12; c = 7; (a + 2) = 15; (b + 2) = 14; c = 7;
$(a - 2)^3 - (b - 2)^3 - c^3 = 11^3 – 10^3 - 7^3 = 1331 – 1000 – 343 = -12;$
$\mathbf{a^3 - b^3 - c^3 = 13^3 – 12^3 – 7^3 = 2197 – 1728 - 343 = 126;}$
$(a + 2)^3 - (b + 2)^3 - c^3 = 15^3 – 14^3 – 7^3 = 3375 – 2744 - 343 = 288;$

Method1: Case 7 n = 4 a = 9 b = 8 c = 7

(a - 2) = 7; (b - 2) = 6 c = 7; a = 9; b = 8; c = 7; (a + 2) = 11; (b + 2) = 10; c = 7;
$(a - 2)^4 - (b - 2)^4 - c^4 = 7^4 – 6^4 - 7^4 = 2401 – 1296 – 2401 = -1296;$
$\mathbf{a^4 - b^4 - c^4 = 9^4 – 8^4 – 7^4 = 6561 – 4096 - 2401 = 64;}$
$(a + 2)^4 - (b + 2)^4 - c^4 = 11^4 – 10^4 – 7^4 = 14641 – 10000 - 2401 = 2240;$

All the above examples clearly demonstrate the turning point nature of $a^n - b^n – c^n$. The value of $\mathbf{a^n - b^n – c^n}$ lies between $(a - 2)^n - (b - 2)^n – c^n$ and $(a + 2)^n - (b + 2)^n – c^n$ for the turning point values of a, b, and c. It also demonstrates that for n = 2 only the turning point is zero. For values of n = 3, and n = 4, the turning point values are always greater than zero. It is possible to get examples for all other values of n and the results will be the same.

The turning point nature of $a^n - b^n – c^n$
Method2 – numerical examples 2

In Method 2, a, b, c are not altered. Here it is found that the turning point value of $a^n – b^n – c^n$ lies between $a^{n-1} – b^{n-1} – c^{n-1}$ and $a^{n+1} – b^{n+1} – c^{n+1}$. Here also, it is found that at the turning point values of a, b, and c, $a^n - b^n – c^n = 0$, for n= 2. For all other values of n, $a^n – b^n – c^n$ is greater than 0 and different. The turning point value of $a^n – b^n – c^n$ depends on the value of n and increases with the value of n. So, for n greater than 2, no two turning point values for different n can be equal or be equal to zero. This fact is proved algebraically in my paper.

Method 2: **Case 1 n = 2 a= 5 b= 4 c= 3**

$\mathbf{a^{(n-1)} = 5, b^{(n-1)} = 4, c^{(n-1)} = 3 \quad a^n = 25, b^n = 16, c^n = 9,}$
$\mathbf{a^{(n+1)} = 125, b^{(n+1)} = 64, c^{(n+1)} = 27}$

$\mathbf{a^{(n-1)} - b^{(n-1)} - c^{(n-1)}} = 5 - 4 - 3 = -2;$
$\mathbf{a^n - b^n - c^n} = 25 - 16 - 9 = 0;$

$a^{(n+1)}$ - $b^{(n+1)}$ - $c^{(n+1)}$ = 125 - 64 - 27 = 34;

Method 2: **Case 2 n = 2 a= 13 b= 12 c= 5**

$a^{(n-1)}$ = 13, $b^{(n-1)}$ = 12, $c^{(n-1)}$ = 5 a^{n} = 169, b^{n} = 144, c^{n} = 25,
$a^{(n+1)}$ = 2167, $b^{(n+1)}$ = 1728, $c^{(n+1)}$ = 125.

$a^{(n-1)}$ - $b^{(n-1)}$ - $c^{(n-1)}$ = 13 - 12 - 5 = -4;
a^{n} - b^{n} - c^{n} = 169 - 144 - 25 = 0;
$a^{(n+1)}$ - $b^{(n+1)}$ - $c^{(n+1)}$ = 2197 - 1728 - 125 = 344;

Method 2: **Case 3 n = 2 a= 25 b= 24 c= 7**

$a^{(n-1)}$ = 25, $b^{(n-1)}$ = 24, $c^{(n-1)}$ = 7 a^{n} = 625, b^{n} = 576, c^{n} = 49,
$a^{(n+1)}$ = 15625, **$b^{(n+1)}$** = 13824, **$c^{(n+1)}$** = 343.

$a^{(n-1)}$ - $b^{(n-1)}$ - $c^{(n-1)}$ = 25 - 24 - 7 = -6;
a^{n} - b^{n} - c^{n} = 625 - 576 - 49 = 0;
$a^{(n+1)}$ - $b^{(n+1)}$ - $c^{(n+1)}$ = 15625 - 13824 - 343 = 1458 ;

Method 2: **Case 4 n = 2 a = 41 b = 40 c = 9**

$a^{(n-1)}$ = 41, $b^{(n-1)}$ = 40, $c^{(n-1)}$ = 9 a^{n} = 1681, b^{n} = 1600, c^{n} = 81,
$a^{(n+1)}$ = 68921, $b^{(n+1)}$ = 64000, $c^{(n+1)}$ = 729.

$a^{(n-1)}$ - $b^{(n-1)}$ - $c^{(n-1)}$ = 41 - 40 - 9 = -8;
a^{n} - b^{n} - c^{n} = 1681 - 1600 - 81 = 0;
$a^{(n+1)}$ - $b^{(n+1)}$ - $c^{(n+1)}$ = 68921 - 64000 - 729 = 4192 ;

Method 2: **Case 5 n = 3 a= 7 b= 6 c= 5**

$a^{(n-1)}$ = 49, $b^{(n-1)}$ = 36, $c^{(n-1)}$ = 25 a^{n} = 343, b^{n} = 216, c^{n} = 125,
$a^{(n+1)}$ = 2401, $b^{(n+1)}$ = 1296, $c^{(n+1)}$ = 625.

$a^{(n-1)}$ - $b^{(n-1)}$ - $c^{(n-1)}$ = 49 - 36 - 25 = -12;
a^{n} - b^{n} - c^{n} = 343 - 216 - 125 = 2;
$a^{(n+1)}$ - $b^{(n+1)}$ - $c^{(n+1)}$ = 2401 - 1296 - 625 = 480 ;

Method 2: **Case 6 n = 3 a = 13 b = 12 c = 7**

$a^{(n-1)}$ = 169, $b^{(n-1)}$ = 144, $c^{(n-1)}$ = 49 a^{n} = 2197, b^{n} = 1728, c^{n} = 343,
$a^{(n+1)}$ = 28561, $b^{(n+1)}$ = 20736, $c^{(n+1)}$ = 2401.

$a^{(n-1)}$ - $b^{(n-1)}$ - $c^{(n-1)}$ = 169 - 144 - 49 = -24;
a^{n} - b^{n} - c^{n} = 2197 - 1728 - 343 = 126;
$a^{(n+1)}$ - $b^{(n+1)}$ - $c^{(n+1)}$ = 28561 - 20736 - 2401 = 5424;

Method 2: **Case 7 n = 4 a = 9 b = 8 c = 7**

$A^{(n-1)}$ = 729, $b^{(n-1)}$ = 512, $c^{(n-1)}$ = 343 a^{n} = 6561, b^{n} = 4096, c^{n} = 2401,
$A^{(n+1)}$ = 59049, $b^{(n+1)}$ = 32768, $c^{(n+1)}$ = 16807.

$A^{(n-1)}$ - $b^{(n-1)}$ - $c^{(n-1)}$ = 729 - 512 - 343 = -126;
a^{n} - b^{n} - c^{n} = 6561 - 4096 - 2401 = 64;
$a^{(n+1)}$ – $b^{(n+1)}$ - $c^{(n+1)}$ = 59049 - 32768 - 16807 = 9474;

International Journal of Advances in Engineering and Management (IJAEM)
Volume 2, Issue 5, pp: 866-879 www.ijaem.net ISSN: 2395-5252

All the above examples clearly demonstrate the turning point nature of $a^n - b^n - c^n$. The value of $\mathbf{a^n - b^n - c^n}$ lies between $a^{(n-1)} - b^{(n-1)} - c^{(n-1)}$ and $a^{n+1)} - b^{(n+1)} - c^{(n+1)}$ for the turning point values of a, b, and c. It also demonstrates that for n = 2 only the turning point value is zero. For values of n = 3, and n =4, the turning point values are always greater than zero. It is possible to get examples for all other values of n and the results will be the same.

NOTES AND REFERENCES

Notes:

In 1995 Andrew Wiles solved the problem and got the Wolfskehl prize amount of $50,000 from Gottingen Academy of Sciences in 1997. Now there is no prize for solving the problem. My aim is to have a record of my research and findings spanning about 38 years. In 2019 I have published one proof in Amazon Kindle Direct Publishing. My second proof is being given to JIMS for making my proof available internationally to present and future mathematicians.

References:

[1]. Edwards, HM (1997). Fermat's Last Theorem. A Genetic Introduction to Algebraic Number Theory. Graduate Texts in Mathematics. **50**. New York: Springer-Verlag.

[2]. KESAVAN NAIRG (Sep 16, 2019) EQUATIONS FOR PYTHAGOREAN TRIPLES & Fermat's last Theorem – a new proof, and Andrew Beal's conjecture – proof for special case. **ISBN-10:** 1693852969 **ISBN-13:** 978-1693852961