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Functional Inequalities About Geometric Means And Arithmetic Means

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ABSTRACT: Functional inequalities are very difficult. Many authors studied functional inequatlities. In this article, we would like to look at some functional inequality problems about arithmetic means and geometric means.

KEYWORDS: Functional inequalities, Arithmetic means, Geometric means.

I. INTRODUCTION

In this paper, we would like to look at some expressions

Arithmetic mean of argument

$$\frac{x+y}{2}, \forall x, y \in \mathbb{R};$$

Geometric mean of argument

$$\sqrt{xy}, \forall x, y \in \mathbb{R}^{+};$$

and

Arithmetic mean of argument

II. ARITHMETIC MEANS AND GEOMETRIC MEANS

Problem 1. Let $\alpha, \beta \in \mathbb{R}$. Determiner all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(1) = \beta; f(t) \geq \alpha t + \beta; \forall t \in \mathbb{R}, \quad (1)$$

and

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}; \forall x, y \in \mathbb{R}. \quad (2)$$

Solution. In (2), let $x = t, y = -t$, then

$$\beta = f(0)$$

$$= f\left(\frac{t+(-t)}{2}\right)$$

$$\frac{f(x)+f(y)}{2}, \forall x, y \in \mathbb{R};$$

Geometric mean of argument

$$\sqrt{f(x)f(y)}, \forall x, y \in \mathbb{R}^{+}.$$

To solve functional inequlity problems, we use substitution method. We usually substitute special values

+) Let $x = t$ such that $f(t)$ appears much in the equation.

+) $x = t, y = v$ interchange to refer $f(t)$ and $f(v)$.

+) Let $f(0) = v, f(1) = v, \ldots$

+) If f is surjection, exist $t: f(t) = 0$ or $t: f(t) = 1$. Choice x, y to destroy $f(g(x,y))$ in the equation. The function has x, we show that it is injective or surjection.

+) To occur $f(x)$.

+) $f(x) = f(y)$ for all x, y ∈ X. Hence $f(x) = const$ for all $x \in X$.

$$\geq \frac{f(t)+f(-t)}{2}$$

$$\geq \frac{(\alpha t+\beta)+(-\alpha t+\beta)}{2}$$

$$= b, \forall t \in \mathbb{R}.$$

Then $f(t) \equiv \alpha t + \beta$. We can check directly $f(t) \equiv \alpha t + \beta$ satisfies (1) and (2).

There for, $f(t) \equiv \alpha t + \beta$.

Corollary 1. Determiner all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(0) = 0; f(t) \geq 0; \forall t \in \mathbb{R}, \quad (3)$$

and

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}; \forall x, y \in \mathbb{R}, (4)$$

is $f(x) \equiv 0$.

Problem 2. Let $\alpha, \beta \in \mathbb{R}^+$. Determiner all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(1) = \alpha; \ f(t) \geq \alpha + \beta \ln t; \forall t \in \mathbb{R}^+; \quad (5)$$

and

$$f\left(\sqrt{xy}\right) \geq \frac{f(x) + f(y)}{2}; \forall x, y \in \mathbb{R}^+. \ (6)$$

Solution. Setting $x = t, y = \frac{1}{t}(t > 0)$, and by (6), we get

$$\begin{aligned} \alpha &= f(1) \\ &= f\left(\sqrt{t \times \frac{1}{t}}\right) \\ &\geq \frac{f(t) + f\left(\frac{1}{t}\right)}{2} \\ &\geq \alpha. \end{aligned}$$

Then $f(t) \equiv \alpha + \beta \ln t$. We can check directly $f(t) \equiv \alpha + \beta \ln t$ satisfies (5) and (6).
There for,

$$f(t) \equiv \alpha + \beta \ln t.$$

Corollary 2. $f(x)$ satisfies

$$f(1) = 1; \ f(t) \geq 1; \forall t \in \mathbb{R}^+; \quad (7)$$

and

$$f\left(\sqrt{xy}\right) \geq \frac{f(x) + f(y)}{2}; \forall x, y \in \mathbb{R}^+. \ (8)$$

is $f(x) \equiv 1$.

Problem 3. Determiner all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(1) = 0; \quad (9)$$

and

$$f\left(\sqrt{xy}\right) \geq \sqrt{\frac{[f(x)]^2 + [f(y)]^2}{2}}; \quad \forall x, y \in \mathbb{R}^+. \ (10)$$

Solution.

By assumption, we have $f(x) \geq 0, \forall x \in \mathbb{R}^+$.

Since $x > 0, y > 0$, we are setting $x = e^u, y = e^v, u, v \in \mathbb{R}$.

Then

$$g(u) \geq 0, \forall u \in \mathbb{R}.$$

In (8), we have

$$g\left(\frac{u+v}{2}\right) \geq \frac{g(u) + g(v)}{2}, \forall u, v \in \mathbb{R}.$$

By Corollary 1, we have $g(u) \equiv 0, \forall u \in \mathbb{R}$.

Then $f(x) \equiv 0$.

We can check all such functions satisfy (7) and (8).
There for,

$$f(t) \equiv 0.$$

Problem 4. Let $k > 1$. Determiner all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(0) = 0; \quad (11)$$

and

$$f\left(\sqrt{xy}\right) \geq \sqrt[k]{\frac{[f(x)]^k + [f(y)]^k}{2}}; \quad \forall x, y \in \mathbb{R}^+. \ (12)$$

Solution.

By assumption, we have $f(x) \geq 0, \forall x \in \mathbb{R}^+$. we have:

$$(12) \Leftrightarrow \left[f\left(\sqrt{xy}\right)\right]^k \geq \frac{[f(x)]^k + [f(y)]^k}{2}; \quad \forall x, y \in \mathbb{R}.$$

Setting

$$g(x) = [f(x)]^k \geq 0,$$

We have

$$g\left(\sqrt{xy}\right) \geq \frac{g(x) + g(y)}{2}; \forall x, y \in \mathbb{R}.$$

By Corollary 1, we have $g(x) \equiv 0$. Then $f(x) \equiv 0$.

We can check all such functions satisfy (11) and (12).

There for, $f\left(t\right)\equiv 0$.

$f\left(t\right)\equiv 0$.

III. CONCLUSION

In this paper, we establish some problems about arithmetic means and geometric means. They are very good for teachers and students.

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