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Functional Inequalities about Harmonic Means and Arithmetic Means

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ABSTRACT: Functional inequalities are very difficult. Many authors studied functional inequatlities. In this paper, we would like to look at some functional inequality problems about harmonic means and arithmetic means.

KEYWORDS: Functional inequalities, Harmonic means, Arithmetic means.

I. INTRODUCTION

In this paper, we would like to look at some expressions

Harmonic means of argument

$$\frac{2xy}{x+y}, \forall x, y \in \mathbb{R}^+;$$

and

Arithmetic means of argument

$$\frac{f(x)+f(y)}{2}, \forall x, y \in \mathbb{R}^+;$$

$$\sqrt{\frac{[f(x)]^2+[f(y)]^2}{2}}, \forall x, y \in \mathbb{R}^+;$$

$$\sqrt[p]{\frac{[f(x)]^p+[f(y)]^p}{2}}, \forall x, y \in \mathbb{R}^+.$$

To solve functional inequlity problems, we use substitution method. We usually substitute special values

+) Let $x = t$ such that $f(t)$ appears much in the equation.

+) $x = t, y = v$ interchange to refer $f(t)$ and $f(v)$.

+) Let $f(0) = v, f(1) = v, \ldots$

+) To occur $f(x)$.

+) $f(x) = f(y)$ for all x, y ∈ X. Hence $f(x) = const$ for all $x \in X$.

II. ARITHMETIC MEANS AND GEOMETRIC MEANS

Problem 1. Let $\alpha, \beta \in \mathbb{R}$. Determiner all functions $f: \mathbb{R} \to \mathbb{R}$ satisfying for arbitrary $\alpha, \beta \in \mathbb{R}$, we have

$$f(1) = \beta; f(t) \geq \alpha t + \beta; \forall t \in \mathbb{R}, \quad (1)$$

and

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}; \forall x, y \in \mathbb{R}. \quad (2)$$

Solution. In (2), let $x = t, y = -t$, then

$$\beta = f(0)$$
$$= f\left(\frac{t+(-t)}{2}\right)$$
$$\geq \frac{f(t)+f(-t)}{2}$$
$$\geq \frac{(\alpha t+\beta)+(-\alpha t+\beta)}{2}$$
$$= b, \forall t \in \mathbb{R}.$$

Then $f(t) \equiv \alpha t + \beta$. We can check directly $f(t) \equiv \alpha t + \beta$ satisfies (1) and (2).

There for, $f(t) \equiv \alpha t + \beta$.

Corollary 1. Determiner all functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$f(0) = 0; f(t) \geq 0; \forall t \in \mathbb{R}, \quad (3)$$

and

$$f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}; \forall x, y \in \mathbb{R}, \quad (4)$$

is $f(x) \equiv 0.$

Problem 2. Determiner all functions $f(t)$ such that

$$f(1)=1; f(t)\geq 1; \forall t \in \mathbb{R}^+; \quad (5)$$

and

$$f\left(\frac{2xy}{x+y}\right) \geq \frac{f(x)+f(y)}{2}; \forall x, y \in \mathbb{R}^+. \quad (6)$$

Solution. Setting $x=\frac{1}{u}, y=\frac{1}{v}$. By (6), we get

$$f\left(\frac{1}{\frac{u+v}{2}}\right) \geq \frac{f\left(\frac{1}{u}\right)+f\left(\frac{1}{v}\right)}{2}, \forall u, v \in \mathbb{R}^+,$$

or

$$g\left(\frac{u+v}{2}\right) \geq \frac{f(u)+f(v)}{2}; \forall u, v \in \mathbb{R}^+, (7)$$

where

$$g(t)=f\left(\frac{1}{t}\right).$$

By (5), we have $g(1)=1,$ and $g(t) \geq 1, \forall t \in \mathbb{R}^+.$

Setting $g(t)=1+h(t),$ we have $h(1)=0,$ and $h(t) \geq 0, \forall t \in \mathbb{R}^+.$

By (7), we get

$$h\left(\frac{u+v}{2}\right) \geq \frac{h(u)+h(v)}{2}; \forall u, v \in \mathbb{R}^+.$$

Let us show that $h(t) \equiv 0$ satisfies $h(1)=0$ and $h(t) \geq 0, \forall t \in \mathbb{R}^+.$

In fact, with $t \in (0,2],$ we have

$$\begin{aligned} 0 &= h(1) \\ &= h\left(\frac{t+(2-t)}{2}\right) \\ &\geq \frac{h(t)+h(2-t)}{2} \\ &\geq 0. \end{aligned}$$

Hence, $h(t)=0,$ for all $t \in (0,2].$ There for,

$$\begin{aligned} 0 &= h\left(\frac{3}{2}\right) \\ &= h\left(\frac{2+1}{2}\right) \\ &\geq \frac{h(2)+h(1)}{2} \\ &= \frac{h(2)}{2} \\ &\geq 0. \end{aligned}$$

Then $h(2)=0.$

So $h(t)=0,$ for all $t \in (0,2].$

Next, with $t \in (0, 2^2],$ we have

$$\begin{aligned} 0 &= h(0) \\ &= h(2) \\ &= h\left(\frac{t+(4-t)}{2}\right) \\ &\geq \frac{h(t)+h(4-t)}{2} \\ &\geq \frac{0+0}{2} \\ &= 0. \end{aligned}$$

Then $h(t)=0,$ for all $t \in (0, 2^2],$ So that

$$0 = h\left(\frac{5}{2}\right)$$

$$=h\left(\frac{2^2+1}{2}\right)$$
$$=\frac{h\left(2^2\right)+h(1)}{2}$$
$$\geq\frac{h\left(2^2\right)}{2}$$
$$\geq 0.$$

Then $h\left(2^2\right)=0$. Conseuently, $h(t)=0$, for all $t\in\left(0,2^2\right]$.

By induction method, let us that $h(t)=0$, for all $t\in\left(0,2^n\right]$, with $n\in\mathbb{N}^*$. Then for $g(t)\equiv 1$. Hence $f(t)\equiv 1$ for all $t\in\mathbb{R}^+$.

We can check directly $f(t)\equiv 1,\forall t\in\mathbb{R}^+$ satisfies (5) and (6).

Problem 3. Determiner all functions $f(t)$ such that

$$f(1)=1;\ f(t)\geq 1;\forall t\in\mathbb{R}^+;\quad (8)$$

and

$$f\left(\frac{2xy}{x+y}\right)\geq\sqrt{\frac{\left[f(x)\right]^2+\left[f(y)\right]^2}{2}};\quad \forall x,y\in\mathbb{R}^+.\ (9)$$

Solution. By assumption, we have $f(x)\geq 0,\forall x\neq 0$. Then for

$$\left[f\left(\frac{2xy}{x+y}\right)\right]^2\geq\frac{\left[f(x)\right]^2+\left[f(y)\right]^2}{2};\quad \forall x,y\in\mathbb{R}^+.$$

$$\Leftrightarrow\left[f\left(\frac{2}{\frac{1}{x}+\frac{1}{y}}\right)\right]^2\geq\frac{\left[f(x)\right]^2+\left[f(y)\right]^2}{2};\quad \forall x,y\in\mathbb{R}^+.$$

Setting $u=\frac{1}{x},v=\frac{1}{y},\forall u,v\in\mathbb{R}^+$, we get

$$\left[f\left(\frac{2}{u+v}\right)\right]^2\geq\frac{\left[f\left(\frac{1}{u}\right)\right]^2+\left[f\left(\frac{1}{v}\right)\right]^2}{2};\quad \forall u,v\in\mathbb{R}^+.$$

Setting,

$$g(u)=\left[f\left(\frac{1}{u}\right)\right]^2\geq 1,\forall u\neq 0,g(1)=1.$$

We have,

$$g\left(\frac{u+v}{2}\right)\geq\frac{g(u)+g(v)}{2};\forall u,v\in\mathbb{R}^+.$$

By Problems 1, we have $g(u)\equiv 1,\forall u\in\mathbb{R}^+$. So, $f(x)=1$.

We can check directly $f(t)\equiv 1,\forall t\in\mathbb{R}^+$ satisfies (8) and (9).

Problem 4. Determiner all functions $f(t)$ such that

$$f(1)=1;\ f(t)\geq 1;\forall t\in\mathbb{R}^+;\quad (10)$$

and

$$f\left(\frac{2xy}{x+y}\right)\geq\sqrt[p]{\frac{\left[f(x)\right]^p+\left[f(y)\right]^p}{2}};\quad \forall x,y\in\mathbb{R}^+.\ (11)$$

Solution. By assumption $f(x)\geq 0,\forall x\neq 0$. We get

$$\left[f\left(\frac{2xy}{x+y}\right)\right]^p\geq\frac{\left[f(x)\right]^p+\left[f(y)\right]^p}{2};\quad \forall x,y\in\mathbb{R}^+.$$

$$\Leftrightarrow\left[f\left(\frac{2}{\frac{1}{x}+\frac{1}{y}}\right)\right]^p\geq\frac{\left[f(x)\right]^p+\left[f(y)\right]^p}{2};\quad \forall x,y\in\mathbb{R}^+.$$

Setting $u=\frac{1}{x},v=\frac{1}{y},\forall u,v\in\mathbb{R}^+$, we get

$$\left[f\left(\frac{2}{u+v}\right)\right]^p \geq \frac{\left[f\left(\frac{1}{u}\right)\right]^p + \left[f\left(\frac{1}{v}\right)\right]^p}{2};$$
$$\forall u,v \in \mathbb{R}^+.$$

Setting,

$$g(u) = \left[f\left(\frac{1}{u}\right)\right]^p \geq 1, \forall u \neq 0, g(1) = 1.$$

We get

$$g\left(\frac{u+v}{2}\right) \geq \frac{g(u)+g(v)}{2}; \forall u,v \in \mathbb{R}^+.$$

By Problem (1), we have $g(u) \equiv 1, \forall u \in \mathbb{R}^+.$

Then $f(x) \equiv 1.$

We can check directly $f(t) \equiv 1, \forall t \in \mathbb{R}^+$ satisfies (10) and (11).

III. CONCLUSION

In this paper, we establish some problems about harmonic means and arithmetic means. They are very good for teachers and students.

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