International Journal of Advances in Engineering and Management (IJAEM)
Volume 5, Issue 3 March 2023, pp: 1487-1493 www.ijaem.net ISSN: 2395-5252

Isomorphism and Homeomorphisms on the Stone Spaces Associated With Boolean Algebras

Monsuru A. Morawo[1], Ismaila O. Ibrahim[2], M. A. Chamalwa[3] and M.L. Danyaro[4]

[1]*SAF Polytechnic, Iseyin, Oyo State, Nigeria*
[2]*Department of Mathematical science, University of Maiduguri, Nigeria*
[3]*Department of Mathematical science, University of Maiduguri, Nigeria*
[4]*Yobe State College of Agricultural Science and Technology, Gujba*

Date of Submission: 20-03-2023 Date of Acceptance: 30-03-2023

ABSTRACT
In this paper, some behaviours of Homeomorphism and Isomorphism were demonstrated on the Stone spaces that associated with Boolean Algebras.
Key words:homeomorphism, homomorphism, isomorphism, Stone space, Boolean Algebra and Spectral.

I. INTRODUCTION AND DEFINITIONS

The original motivation for developing topology was a wish to generalize ideas in geometry such as shape and distance. The notion that the topology can illuminate algebra was resisted by such luminaries as Birkhoff, whose once, when asked about the utilization of topological notions in algebra, allegedly responded, "I don't consider this algebra, but this does not mean that algebraists cannot do it" [1].

Nevertheless, in 1936, Marshall Stone published a paper [2] showing an unexpected equivalence relation between Boolean algebras and certain topological spaces, which is known as Stone spaces. In this paper, some behaviour of homeomorphisms and isomorphism were demonstrated on the Stone spaces that associated with Boolean Algebras.

Definition 1.1: [3] A lattice is a partially order set X such that every finite subset of X has an intersecting point. A lattice is complete if it has an intersection between arbitrary also, not just finite sets. Also, it is distributive if it allows the distribute property on it. i.e. $\forall x, y, z \in X$. $x \wedge (y \vee z) = (x \wedge y) \vee (a \wedge z)$.

Example 1.1: [5]
(a) The symbols $\wedge$ and $\vee$ represent the set operations intersection $(\cap)$ and union $(\cup)$. i.e. if we consider arbitrary nonempty set X and P(X) denotes its power set. By taking $0 = \emptyset$ and $X = 1$. We let the symbol $(\wedge)$ stand for intersection $(\cap)$, $(\vee)$ to be unions $(\cup)$ and $-$ denontes the complement. Then for $X_1, X_2 \in P(X)$, we say $X_1 \leq X_2$, if $X_2 \subseteq B$. Therefore the power set P(X) is a complete distributive lattice.

Definition 1.2: [1] If A is a distributive lattice and if for every $x \in A\ \exists\ -x \in A$ such that $x \wedge (-x) = 0$ and $x \vee (-x) = 1$. Then $-x$ denotes complement of x.

Definition 1.3: [3] Every distributive lattice A for which every element $x \in A$ has complement $-x \in A$ is called Boolean algebra.

Proposition 1.1: [3] In every Boolean algebra A, the complement $-x \in A$ is unique.

Definition 1.4: [6] A topological space X is profinite if it is homeomorphic to a projective limit of finite discrete space.

Proposition 1.2: [2] For a Boolean Algebra A, the following properties holds.

(a) $\vee$ and $\wedge$ are idempotent, commutative and associative.
(b) $\vee$ and $\wedge$ are distributive. i.e. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
(c) Absorption property holds for $\vee$ and $\wedge$ i.e. $x \vee (x \wedge y) = x = x \wedge (x \vee y)$.
(d) Every element is its own double complement, i.e $-(-x) = x$
(e) $-(x \vee y) = -x \wedge -y, -(x \wedge y) = -x \vee -y$ (De Morgans property)
(f) As shown in the **definition 1.2** above, $x \vee (-x) = 1$ and $x \wedge (-x) = 0$

(g) $x \lor 0 = x, x \land 1 = x, x \lor 1 = 1$ and $x \land 0 = 0$

Example 1.2: [5] The following are example of Boolean algebras
(a) Any complete lattice is a Boolean algebra
(b) The set $\{0,1\}$ is a Boolean algebra, for $0 \leq 1$.
(c) For every set A, the power set P(A) is a Boolean algebra with respect to the following operations $\lor = \cup$, $\land = \cap$, $0 \neq \emptyset$, $1 = X$ and $-X = Y \setminus X$.
(d) If X is a topological space. $A = \mathrm{Cl}\, P(X)$ equiped with operations mentioned in **(c)** above is a Boolean algebra

Definition 1.5: [7] Let A be a Boolean algebra. A subset $I \subseteq A$ is called an Ideal if the following properties holds on I.
(a) $0 \in I$ (b) If $x, y \in I$, then $x \lor y \in I$ (c) If $y = I$ and $x \leq y$, then $x \in I$.
Definition 1.6: [8] An ideal I is proper ideal if $1 \notin I$. A proper ideal is maximal. If it is not properply contained in another proper ideal. i.e. If I is a proper ideal, then there does not exist $x \in I$ and $-x \in I$, provided that the properties **(b)** above implies $1 = x \lor -x \in I$, which is contrary to properness of an ideal.
Example 1.3: [1] In a given topological space X, the lattice of open sets ordered by inclusion which is denoted as $\Omega(X)$ is an ideal in power set P(X) of topological space X.
Definition 1.7: [7] Let X and Y be Boolean algebras A Boolean Homomorphism is a mapping $g: X \longrightarrow Y$such that $\forall\, x, y \in X$, the following properties are satisfied: (a) $g(x \lor y) = g(x) \lor g(y)$, (b) $g(x \land y) = g(x) \land g(y)$ and (c) $g(-P) = -g(P)$. Where the operations on the left side of each properties above are in X while those on the right sides are in Y. The property (c) above shows that a Boolean Homomorphism $g: X \rightarrow Y$ is odd. If a homomorphism is onto, then is called epimorphism, if it is an 1 – 1 correspondence (bijective), is called isomorphism. Then, if X and Y are Boolean algebras and there is Isomorphism between them, we say X and Y are isomorphism to each other.
Definition 1.8: [3] A topological space X is τ_2 space (Hausdoff), if for any distinct point $x, y \in X$, there exist two disjoint open sets A and B such that $x \in A$ and $y \in B$.
Definition 1.9: [3] A topological space X is compact, if every open cover of X contain a finite subcover.
Definition 1.10: [4] A $\tau_2 -$ space (Hausdorff) is totally disconnected if every open set is the union of the clopen sets it contains.

Definition 1.11: [5] A topological space X is called a stone space if it is τ_2, compact and totally disconnected. If B is a Boolean algebra, S(B) will denotes the stone space associated with Boolean algebra A.

II. RESULTS

In this section, we are going to make use of some definition and other discussions in section one to demonstrate some roles of isomorphism and homeomorphisms in the stone spaces that associated with Boolean algebras.
Lemma 2.1: [2] Suppose A is a Boolean algebra and an ideal $I \subset A$ is maximal iff for every $a \in A$, either $a \in I$ or $-a \in I$ both not both.
Theorem 2.1: (Maximal ideal theorem) Every proper ideal is contained in a maximal ideal.
Proof: Since the set of ideals is poset (partially order) under inclusion relation $\subseteq$. Then, it is easy to check that if $\mathcal{F}$ is a totally ordered family of proper ideal, certainly, $\cup\mathcal{F}$ is a proper ideal. Therefore, the claim follow from the fact that if X is a nonempty partially ordered set in which every totally ordered subset has an upper bound. Then X contains at least one maximal element.
Definition 2.1: [9] Suppose A is a Boolean algebra, then $\mathrm{spec}(A)$ is called a stone space associated with Boolean algebra A. Where **spec** denotes the spectrum.
Definition 2.2: [10] Suppose X is a stone space, then the dual algebra of X is the class of open and closed set in X. The dual algebra of a stone space X is denoted as $\mathrm{clop}(X)$.
Theorem 2.2: If P, Q are Boolean algebras, then there is isomorphism between P and Q iff there is isomorphism between $\mathrm{spec}(P)$ and $\mathrm{spec}(Q)$.
Proof: From **definition 2.1** above, it is obvious that isomorphism between P and Q implies isomorphism between $\mathrm{spec}(P)$ and $\mathrm{spec}(Q)$ i.e. $P \cong Q \Longrightarrow Clop(P) \cong clop(Q)$.
Conversely, suppose $\mathrm{spec}(P) \cong spec(Q)$. By follow the trivial implication, $P \cong Q \Longrightarrow Clop(P) \cong clop(Q)$ (from **definition 2.2** above) and from the fact that $clop(spec(X)) \cong X$, we have $P \cong clop(spec(P)) \cong clop(spec(Q)) \cong Q$, which ends the proof.

Theorem 2.3: Suppose A is a finite Boolean algebra. Then, the following are equivalents:
(a) $A \cong P(A) = \{0,1\}^{\#A}$ and has A has 2^n elements for some $n \in \mathbb{N}$
(b) There is a homomorphism $g: A \longrightarrow 2$ such that $g(2^n) = 1$ where 2^n is a nonzero element of Boolean algebra A.

International Journal of Advances in Engineering and Management (IJAEM)
Volume 5, Issue 3 March 2023, pp: 1487-1493 www.ijaem.net ISSN: 2395-5252

(c) If there exist $S(A) \subset \{0,1\}^{\#A}$ of homeomorphism $g: A \longrightarrow 2$. Then $S(A)$ is a Stone space.

Proof: Suppose B is a finite Boolean algebra.

(a) If we let $A = spec(B)$. Then, by finiteness of B and stone duality, the stone space A must have finitely many clopens. This mean that A is direct sum of connected subspaces. Provided that the Stone space A is totally separated, the latter are all set of single element, thus A is discrete, and $\text{clop}(A) = P(A)$. Then, by compactness, A must be finite. This shows that the Boolean algebra of clopens of a finite set A is isomorphic to $\{0,1\}^{\#A}$.

(b) If I is a principal ideal that generated by (-2^n). Then by **theorem 2.1** above, there exist some maximal ideal M such that $(-2^n) \in M$. By **Lemma 2.1** above, M does not contains 2^n. Then due to the fact that every proper ideal is the kernel of some epimorphism between given Boolean algebras, there are some homomorphism $g: A \longrightarrow 2$ whose has kernel M. Provided that $2^n \notin M, g(2^n) = 1$.

(c) For a Boolean algebra A, let $m \in A$, then for every $n \in \{0,1\}^{\#A}$, the singleton set $\{n_m\}$ is open in $\{0,1\}$, since the set $\{0,1\}$ is a discrete topology, and the following sets $\{n \in$ $0,1 m: nm=1$ and $n \in 0,1 m: nm=0$ are open in $\{0,1\}^{\#A}$. Therefore, the value of n_m must depends on n continuously. Now, suppose $P: X \longrightarrow Y$ and $g: X \longrightarrow Y$ are continuous function into a τ_2 – space Y, the set $\{n: P(n) =$ gn is closed. Therefore, for $2n \in A$, the set $\{n: n(-2^n) = -n(2^n)\}$ is closed in $\{0,1\}^{\#A}$. Hence, the intersection of all complement preserving functions in $\{0,1\}^{\#A}$. Similarly, the set of every functions in $\{0,1\}^{\#A}$ that preserves meets and joins are closed subset of $\{0,1\}^{\#A}$.

The intersection of these three sets gives the set of functions that preserves meets($\wedge$), joins($\vee$) and complements ($-$), so, the set K of homomorphism $2^n: A \longrightarrow 2$. The intersection of closed subset is closed, for this, K is closed and hence is compact due to the fact that $\{0,1\}^{\#A}$ is τ_2 and totally disconnected, then, is a Stone space.

Theorem 2.4: Suppose A and B are Stone spaces, then A is isomorphic to B if and only if $\text{clop}(A)$ is isomorphic to $\text{clop}(B)$.

Proof: If part of this theorem is trivial. Now, suppose $Clop(A) \cong Clop(B)$, this gives leads to $spec(clop(A)) \cong spec(clop(B))$. By **theorem 2.2** above, we have $A \cong spec(clop(A)) \cong spec(clop(B)) \cong B$. This ends the proof.

Theorem 2.5: (Stone Representation theorem for Boolean Algebras). Every Boolean algebra is isomorphic to the dual algebra of its associated Stone space. i.e. if B is Boolean algebra and $spec(B)$ is a stone space, then $B \cong clop(spec(B))$ for every B.

Proof: Suppose $spec(B)$ is the set of all maximal ideals in B such that $spec(B) \subseteq P(B)$. Then, for every $x \in B$, we have $T(x) = \{K \in spec(B): x \notin$ K. Then, we have $T0=\emptyset$ and $T1$ = stone space $spec(B)$.

Suppose I is any ideal, then by **(b)** and **(c)** in **definition 1.4** above, we have $x \vee y \in I$, if and only if $x \in I$ and $y \in I$. Suppose M is a maximal ideal. By **Lemma 1.1** above, we can see that $x \wedge y \in M$, if and only if $x \in M$ or $y \in M$. So, we have the following:

$T(x) \cup T(y) = \{M \in spec(B): x \notin M \text{ or } y \notin M\} =$
$\{M \in spec(B): -(x \in M \text{ and } y \in M)\} =$
$\{M \in spec(B): -((x \vee y) \in M)\} = T(x \vee y)$ ------------------------------------ **(i)**

$T(x) \cap T(y) = \{M \in spec(B): x \notin M \text{ and } y \notin$
$M = M \in specB: -x \in M$ or
$y \in M = M \in specB: -x \wedge y \in M = Tx \wedge y$ ------------------------------------ **(ii)**

Therefore, the family $\mathcal{F} = \{T(x)\}_{x \in B}$ is closed under intersection and certainly,
$T(1) = spec(B) \subset \{T(x)\}_{x \in B} = \mathcal{F}$. Due to the fact that, if X is a set, $\mathcal{F} \subseteq P(x)$ and
$\tau = \{\bigcup_{i \in I} U_i : I \text{ is a set}, \ U_i \in \mathcal{F} \ \forall \ i \in I\}$. Then τ is a topology if and only if (i) Every $x \in X$ is contained in some $U \in \mathcal{F}$ and (ii) for every $P, Q \in \mathcal{F}, P \cap Q = \bigcup\{N \in \mathcal{F}: N \subseteq P \cap Q\}$, the family of all unions of elements of $\mathcal{F}$ is a topology on the space $spec(B)$, which is called the Stone topology with $\mathcal{F}$ as its base. By **Lemma 1.1** above, for $x \in B$ and $M \in spec(B)$, we have any of $x \in M$ or $(-x) \in M$ but not both. Therefore,
$T(-x) = \{M \in spec(B): (-x) \notin M\} =$
$\{M \in spec(B): x \in M\} = spec(B) - T(x)$ ------------------------------------ **(iii)**

(iii) above implies that all elements of base $\mathcal{F}$ is both open and closed (clopen), therefore, $\mathcal{F} \subseteq clop(spec(B))$, this means, $\mathcal{F}$ serves as a base of a space $spec(B)$ and $\mathcal{F}$ contains clopens, for this, the stone space $spec(B)$ is zero – dimensional.

From **(i)** and **(ii)** above, if there is $X, Y \in M$ such that if $X, Y \in spec(B)$ with $X \neq Y$, the inclusion $X \subseteq Y$ is not possible, provided that X is maximal, then there exist $x \in (X - Y)$, $(-x) \notin X$, $x \in Y$, thus $(-x) \in (X - Y)$. Now, $X \in T(-x)$ and
$Y \in T(x) = spec(B) - T(-x)$, provided that family of T's are open, the stone topology $(spec(B), \tau)$ is $\tau_2 - space$.

International Journal of Advances in Engineering and Management (IJAEM)
Volume 5, Issue 3 March 2023, pp: 1487-1493 www.ijaem.net ISSN: 2395-5252

Let O be a cover of the space $spec(B)$ by element of base $\mathcal{F}$, then for $\{a_k \in B\}_{k\in K}$ such that $spec(B) = \bigcup_{k\subset K} O(a_k)$. Now, let I be the set of $x \in B$ for which there are finitely $k_1, k_2, \dots, k_n$ such that $x \le a_{k_1} \vee \dots \vee a_{k_n}$. It is easy now to verify that I is an ideal. Suppose I is proper and there is maximal ideal M such that $I \subseteq M$, then, we have $a_k \in I \subseteq M$ for every $k \in K$, provided that $\{T(a_k)\}_{k\in K}$ is a cover of the space $spec(B)$, then, we have $M \in T(a_k)$ for some $k \in K \Rightarrow a_k$ does not exist in maximal ideal M, which is contrary to our choice. Thus $I = B$. then $i \in I$, so that there are $k_1, k_2, \dots, k_n \in K$ such that $a_{k_1} \vee a_{k_2} \vee \dots \vee a_{k_n} \Rightarrow T(a_{k_1}) \cup T(a_{k_2}) \cup \dots \cup T(a_{k_n}) = T(a_{k_1} \vee a_{k_2} \dots \vee a_{k_n}) = T(1) = spec(B)$.

Therefore, every cover $O(a_k)$ of the space $spec(B)$ by elements of the base $\mathcal{F}$ possess a finite subcover, therefore, the space $spec(B)$ is compact. Hence is a stone space.

Let $A \subseteq spec(B)$ be open and closed (clopen). Since A is open and there are element $x \in \mathcal{F}$, $A = \cup x$. Also, since A is closed, then it is compact, thus

$A = T(x_1) \cup \dots \cup T(x_n) = T(x_1 \vee x_2 \vee \dots \vee x_n) \in \mathcal{F} \Rightarrow \mathcal{F} \subseteq Clop(spec(B))$ exhaust $clop(spec(B))$.

By consider equation **(i)**, **(ii)**, $T(1) = spec(B)$, $T(0) = \emptyset$, then the map $g: B \to clop(spec(B))$, $x \to T(x)$ is a homomorphism of Boolean algebras whose its onto is obvious, provided that

$\mathcal{F} = \{T(x): x \in B\} = clop(spec(B))$. Suppose r is nonzero element, then $I = \{y \le (-x)\}$ is a proper ideal. Since M is maximal ideal that contains an ideal I, we have $-x \in I \subseteq M$ as shown above, then $x \notin M$ and therefore $M \in T(x)$. Thus x is nonzero element implies that $T(x)$ is nonempty. Now, suppose $x \neq y$. Then $x \wedge (-y)$ is nonzero or $y \wedge (-x)$ is also nonzero. Then $T(x) - T(-y) = T(x) \cap T(-y) = T(x \wedge (-y))$ is nonempty,

thus, $T(x) \neq T(y)$. If $y \wedge (-x)$ is nonzero, the same conclusion holds. Then the mapping $g: B \to clop(spec(B))$ is one – one. Therefore, g is an isomorphism of Boolean algebras.

Theorem 2.6: Suppose X is a topological space and there is a mapping $G: X \to P(clop(X))$ defined by $G: X \to P(clop(X)), x \longrightarrow \{C \in clop(X): x \notin C$ -------------------------------- **(iv)**

The following statements are equivalent

(i) G is $1-1$ iff X is totally separated

(ii) $G(x)$ is maximal ideal in $clop(X)$ for every $x \in X$, thus $G: X \to spec(clop(X))$.

(iii) $G: X \to spec(clop(X))$ is continuous.

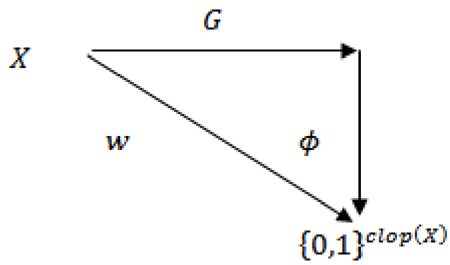
(iv) $Cl(G(X)) = spec(clop(X))$.

(v) If X is compact, then the mapping $G: X \to spec(clop(X))$ is onto and $G: X \to spec(clop(X))$ is homeomorphism iff X is a Stone space.

From the knowledge of **theorem 2.6** above, we shall establish the following results

Lemma 2.2: Given a topological space X, we define the mappings $w: X \to \{0,1\}^{clop\,(X)}$, $x \to \Pi_{R\in clop\,(X)} \chi_A(x)$. Then w is one – one, if X is totally separated.

Theorem 2.7: For every topological space X defined the mapping $\phi: P(clop(X)) \to \{0,1\}^{clop\,(X)}$ by $\phi(A)(R) = 1 \setminus X_A(R)$. Then G is $1-1$ correspondence and the diagram below commutes.



Proof: we shall follow **lemma 2.2** above to provide a proof for this theorem. Let $P(clop(X))$ associated with the function $\{0,1\}^{clop\,(X)}$ and subsets S of X through $S \to X_S$ and $g \to g^{-1}(1)$. From **(iv)** above, we have $G: x \to \{R \in$ $clopX:x\notin R = R\in clopX:\chi Rx=0$ --- **(v)**

By **(v)** above, "we see that" there is occurrence of negation, whereas we don't have any in $w(x)$, which is defined as $w = \{R \to \chi_R(x)\}$. From this, we see that ϕ is $1-1$ correspondence and the diagram above commutes. Next, we need to provide a proof for the continuity of ϕ, to do this, we shall prove that $\phi^{-1}(k)$ is open for elements of the canonical subbase k_π with respect to the product topology on the function $\{0,1\}^{clop\,(X)}$. Let $S = V_R^{-1}(t)$ and $R \in clop(X)$ as shown above and t is in the function $\{0,1\}$. We can now have the function

$V_R^{-1} = \{r: clop(X) \to \{0,1\}: r(R)\ is\ unity\}$.

Therefore, the function $(map)\phi_X^{-1}(V_R^{-1}(1)) = \{A \subseteq clop(X): R \notin A\} = \cup(R)$ ---------------------------------- **(vi)**

By Stone topology, we see the **(vi)** above is open. Also, $\phi_X^{-1}(V_R^{-1}(0)) = \{A \subseteq clop(X): R \in A\} = spec(clop(X)) - \cup(R) = \cup(X - R)$ (From equation **(iii)** of **theorem 2.4** above). Therefore, the restriction of ϕ in the Stone space $spec(clop(X))$ is continuous.

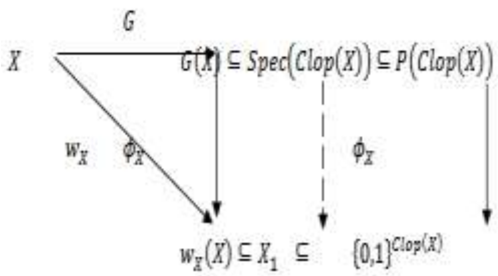
The next results relates the function $G: X \to spec(clop(X))$ to compact Hausdorff spaces X_1

and X_2, where $X_1 < X_2$. Also, for every topological space X which give rise to a Stone space $\Omega_X = \Omega(clop(X))$, we will also establish a direct relation between topological space X and the Stone space Ω_X.

Theorem 2.8: Given a compact Hausdorff topological space X. Then the following are equivalent:
(i) The space $(X-\sim)$ is a Stone space.
(ii) The Stone space $\Omega(clopen(X))$ is homeomorphic to the Stone space $(X-\sim)$.
Proof: Here, we shall prove that for every closed subspace Y of X, the $\sim$ saturation $Y^{\sim}$ is closed. For this reason, there exist min X such that $m \in (X - Y^{\sim})$. Then for all $n \in Y$, $m \nsim n$. Note that, for a compact Hausdorff space X, whenever $m \nsim n$, there must be a clopen set Y in X such that $n \in Y$ but $m \notin Y$. Thus, for every $n \in Y$, a clopen C_n is closed in X such that $n \in C_n$. Now, $Y \subseteq X$ is closed, thus compact. Then, the open cover $\{C_n\}_{n\in Y}$ of Y contain a finite subcover. Then there exist $y_1, y_2, \dots y_n \in Y$ such that $S = Y \subseteq \bigcup_{i=1}^{k} C_n$ -------------- **(*)**.
From **(*)** above, as S is a finite union of clopen, then is closed. Therefore, the space $(X - S)$ is open and also, with fact that $m \in (X - S) \subseteq (X - Y^{\sim})$, $Y^{\sim}$ is closed, then compact, which shows that $(X-\sim)$ is a Stone space.
(ii) provided that X is compact Hausdorff, the connected component $C(x)$ equals the quasi – component $Q(x)$. Therefore, $x \sim y$ iff both x and y are contained in the common clopens. Certainly, the set of clopens is in canonical bijection (1 – 1 correspondence) to the Stone space $(X-\sim)$. This ends the proof.

Theorem 2.9: Suppose X is completely regular. Then, there exists a homeomorphism $\Omega(clopen(X)) \cong (X_2-\sim)$. Where $\sim$ is a connected relation defined on a maximal compact Hausdorff space X_2.
Proof: Due to the fact that for a completely regular space X. Every clopen C in X is of the form $S \cap X$ for a unique clopen $S \subseteq X_2$ and X_2 is connected iff X is connected. For this, there exist 1 – 1 correspondence between the clopens C in X, then the idempotents
$i^2 = i \in C_b(X, R) \cong C(X_2, R)$ and clopen in X_2. Provided that every isomorphism between commutative algebras provides an isomorphism of the respective Boolean algebras of idempotents. There is a mapping $Clop(X_2) \longrightarrow Clop(X)$, $S \rightarrow S \cap X$ which is certainly an isomorphism of Boolean algebras. Then, $\Omega(clop(X_2)) \cong (X_2 - \sim)$. Which ends the proof.

Theorem 2.10: Let X be a topological space. Then, the following statements holds
(i) The mapping $\phi_X \rightarrow Spec(clopen(X))$ is a homeomorphisms
$Spec(clop(X)) \rightarrow X_1 \subseteq \{0,1\}^{clop\,(X)}$.
(ii) If X is strongly zero-dimensional, then the space $(Spec(Clop(X), G))$ is X_1 and the space $(Spec(Clop(X), G)) \cong (X_1, w_X)$ in $C(X)$.
(iii) If X is strongly zero-dimensional, then the space $(Spec(Clop(X), G)) \cong (X_1, w_X) \cong (X_2, w_X)$ and
(iv) If X is infinite, then $(U\mathcal{F}(X), F) \cong (Spec(P(X)), G) \cong (X_1, w_X) \cong (X_2, w_X)$.
Proof:
(i) By considering the diagram below;



From, **theorem 2.5** above, the triangle in the inclusion above is commutes. Then, the map $\phi_X \longrightarrow Spec(Clop(X))$ is a homeomorphism of the Stone space $Spec(Clop(X))$ onto its image, which is also closed in the function $\{0,1\}^{Clop\,(X)}$. Thus $X_1 = Cl(X_1(X))$ is in the map $\phi_X \longrightarrow Spec(Clop(X))$. Now, provided that $Cl(G(X)) = Spec(Clop(X))$ and the map $\phi_X \longrightarrow G(X) = w_X(X)$, then $Cl(w_X(X))$ contains the map $\phi_X \longrightarrow Spec(Clop(X))$, this implies that $\phi_X \longrightarrow Spec(Clop(X)) = X_1$.
(ii) Suppose X is zero dimensional, then the map $w_X: X \rightarrow X_1$ is an embedding, thus, the space (X_1, w_X) is τ_2– compactification of X. Then, by the commutativity of the triangle in the inclusion figure above and the homeomorphism $\phi_X \longrightarrow Spec(Clop(X)) \rightarrow X_1$. This evidence is follow by using the fact that $G: X \rightarrow Spec(Clop(X))$ is continuous for a given topological space X_1.
(iii) From the fact that if X is a strong zero-dimensional space, $X_2 \cong X_1$ in $C(X)$, and X_2 is a Stone space, for $X_1 < X_2$ and X_2 is called the maximal compactification of X and X_1 is τ_2 –

compactification of X. By consider (ii) above, we see that
$\big(Spec(Clop(X),G)\big) \cong (X_1, w_X), \Rightarrow$
$\big(Spec(Clop(X),G)\big) \cong (X_1, w_X) \cong (X_2, w_X)$.
(iv) Provided that infinite space is strongly zero – dimensional and $Clop(X) = P(X)$, by **(iii)** above, we shall provides a $1-1$ correspondence $K: Spec\big(P(X)\big) \to U\mathcal{F}(X)$. Note that a maximal ideal I_{max} in $Spec\big(P(X)\big)$ is a family $I_{max} \subseteq P(X)$ which has the empty set is closed under union and subsets and $X \nsubseteq I_{max}$ and is maximal. Therefore, the family $\mathcal{F} = \{(X - C): C \in I_{max}\}$ is an ultrafilter, by definition of filter and ultrafilter, there exist occurrence of $1-1$ correspondence $K: I_{max} \to U\mathcal{F}(X)$. This ens the proof.

Lemma 2.3: Suppose A is a Boolean algebra. The clopen subset $B \subseteq A$ is a Boolean algebra.

Proof: Given that A is a Boolean space and $B(A)$ denotes the set of all clopen subsets of A. Notice that $\emptyset, A$ are in $B(A)$. Suppose B, C are in $B(A)$, then $B \cap C$ and $B \cup C$ must belongs to $B(A)$. Therefore, if $C \in B(A)$, certainly, $Cl(C) \in B(A)$.

Lemma 2.4: Suppose A is a Boolean algebra. Then, the following properties hold:
(a) $T_0 = \emptyset$
(b) $T_1 = \tau_A$
(c) $T_x \cap T_y = \cup_{x \wedge y}$
(d) $\cup_x \cup \cup_y = \cup_{x \vee y}$ and
(e) $\cup_{(-x)} = Cl(\cup_x)$

From the **Lemma 2.3** above, we can defined a topology τ on $spec(A)$ whose open sets are union of sets of the form $\cup_x$. Then the topological space $(Spec(A), \tau)$ denotes the Stone space of Boolean algebra A.

Theorem 2.11: For every Boolean algebra A, the Stone topological space $(Spec(A), \tau)$ is Boolean space.

Proof: Provided that the topological space $(Spec(A), \tau)$ is zero – dimensional follow by **(e) of Lemma 2.4** above. Suppose X and Y are two ultrafilters such that $X \neq Y$. There must be an element $x \in (X - Y)$. Note that $X \in T_x$, $Y \in T_{-x}$ and $T_x \cap T_{-x} = \emptyset$. So, the space $(Spec(A), \tau)$ is τ_2 – space. Next, we shall prove that the topological space $(Spec(A), \tau)$ is compact by letting $K = \{T_i : i \in I\}$ be a cover of topological space $(Spec(A), \tau)$. Assume that there does not exist finite subset of K that covers $(Spec(A), \tau)$. Then, for every $t_1, t_2, \ldots, t_n \in I$, we have $T_{x_1} \cup T_{x_2} \cup \ldots \cup T_{x_n} \neq (Spec(A), \tau)$. This implies that $x_1 \vee x_2 \vee \ldots \vee x_n \neq 1$ and $(-x_1) \wedge (-x_2) \wedge \ldots \wedge (-x_n) \neq 0$. Therefore, the set $(-I) = \{(-x): x \in I\}$ has the finite intersection property. By prime ideal theorem for Boolean algebra, there exist an ultrafilter U such that $(-I)$ is contained in U. Therefore, $U \in T_x$ for some $x \in I$ and $x, (-x) \in U$ and is contrary to our aim.
In the next result, we provides a proof for the other version of Stone representation theorem.

Theorem 2.12: Suppose A is Boolean algebra. Then, the following statements holds.
(a) $A \cong Clop\big(Spec(A)\big)$
(b) $A \cong Spec\big(Clop(A)\big)$

Proof: To provide a proof for these, we shall consider **theorems 2.2, 2.4** and **2.5** above.
(a) Suppose there is a mapping $f: A \to Clop\big(Spec(A)\big)$ and $x \to T_x$. By follow **Lemma 2.4** above, the map $f: A \to Clop\big(Spec(A)\big)$ is a homomorphism of A. Due to the fact that if x, $y \in A$, where $x \neq y$ and they are nonzero. Then, there exist an ultrafilter that contains either x or y but not both, for this f is one to one and also, by the fact that every closed subset of compact space is compact, it is compact.
(b) Suppose $n \in A$, then, there is a neighbourhood N_n which is the set of all clopen set that contain n. It is not difficult to verify that N_n is an ultrafilter in $Spec(A)$ and therefore $N_n \in Spec\big(Clop(A)\big)$. Let us defined the mapping $g: A \to Spec\big(Clop(A)\big)$ by $n \to N_n$, which is the set of all clopen set that has n as element. Provided that A and $Spec(A)$ are compact Hausdorff space, we need to prove that the mapping g is homeomorphism. Suppose there exist two sets of clopen set that contains two disjoint elements, say N_n and N_m and if $N_n = N_m$, by the fact that A is τ_2 - space, there exist two disjoint open neighbourhood P and Q such that $n \in P$ and $m \in Q$. Since X is zero dimensional, then we can consider both P and Q as clopen for which both N_n and N_m are not equal. Then the mapping $g: A \to Spec\big(Clop(A)\big)$ is one – one. Further, let U be an ultrafilter that contains all clopen subsets of compact set Clop(A), we need to show that these clopen subsets are closed. Provided that U is a filter with finite intersection property, then there exist an element n in the $\cap_{i=1}^{k} t_i$ for every $t \in U$. Thus, U is contained in N_n. But U is an ultrafilter, so, $U = N_n$. Next, we shall show that the mapping $g: A \to Spec\big(Clopen(A)\big)$ is continuous. If $E \subseteq Spec\big(Clop(A)\big)$ and open in $Spec\big(Clop(A)\big)$. Then, $E = \cup_{i=1}^{k} O_i$, where O_i is the basic open sets in E that are clopen. Then, we have E_{O_i} as the representation of open set that are clopen in the

International Journal of Advances in Engineering and Management (IJAEM)
Volume 5, Issue 3 March 2023, pp: 1487-1493 www.ijaem.net ISSN: 2395-5252

space $\text{Spec}(\text{Clop}(A))$. Therefore, $N_n \in E_{0_i}$ if and only if $n \in 0_i$. Therefore, $g^{-1}(E_{0_i}) = A$. Which ends the proof.

III. CONCLUSION

In this work, we realised that:

(i) If X is a Stone space, then the space $\Omega(\text{Clop}(X)) \cong X$

(ii) If X is compact Hausdorff space, then,$\Omega(\text{Clop}(X))$ is homeomorphic to the space $(X{-}{\sim})$ and $\Omega(\text{Clop}(X)) \cong \pi_C(X) = (X{-}{\sim})A \cong \text{Clop}(\text{spec}(A))$ and $A \cong \text{Spec}(\text{Clop}(A))$.

(iii) If X is $\tau_{3.5}$, then $\Omega(\text{Clop}(X)) \cong \pi_C(X_2) = (X_2{-}{\sim})$, where X_2 is a Stone-Čech compactification (maximal compactification).

(iv) If X is strongly zero – dimensional, then $\Omega(\text{Clop}(X)) \cong X_2$ and

(v) Every Boolean space is homeomorphic to the Stone space.

REFERENCES

[1]. Peter T. Johnstone “Stone Spaces” Cambridge University press. Cambridge 1982

[2]. Mashall H. Stone. “The theory of Representations of Boolean Algebras” Transactions of the American Mathematical Society 40 (Jul. 1936), 37 – 111.

[3]. Monsuru. A. Morawo “Topology Of Power Set”.International Journal of Scientic and Engineering Research (IJSER). ISSN 2229 – 5518.

[4]. Ahmadu Kiltho and Monsuru A. Morawo “On some topological properties on a certain hyperspace topology”.IOSR Journal of Mathematics (IOSR - JM). Volume 17, Issue 1.Jan – Feb 2021. 34 – 38.

[5]. I. Fleisher “Priestley’s duality from Stone’s “. Adu. Appl. Math. (2000) 233 – 238.

[6]. G. Bezhanishuili, D. Gabelaia, M. Jibladze and P. J. Morandi “Profinite Topological spaces” Theory and Applications of categories, Vol 30, No. 53 (2015). 1841 – 1863.

[7]. William Hanf “On some fundamental problems concerning isomorphism of Boolean Algebras” Math. Scand. 5 (1957), 205 – 217.

[8]. Elliott Mendelson “Boolean Algebra and Switching Circuits” Schaum’s Outline series McGRAW – HILL BOOK.

[9]. Mirte Dekkers “Stone duality” An application in theory of formal languages” A master thesis. December 2008.

[10]. Amy Schardijn “An introduction to Boolean Algebras” Electronic Theses, project and Dissertations. 421 (2016).