International Journal of Advances in Engineering and Management (IJAEM)
Volume 4, Issue 8 Aug. 2022, pp: 1098-1100 www.ijaem.net ISSN: 2395-5252

On Generalized Rational Functions - 1

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Submitted: 10-08-2022 Revised: 20-08-2022 Accepted: 22-08-2022

ABSTRACT

A normalized function f analytic in the open unit disc around the origin and nonvanishing outside the origin can be expressed in the form z/f (z) where g (z) has Taylor coefficients b_n's. Necessary and sufficient conditions in terms of b_n's are derived for some classes of analytic functions.

INTRODUCTION

Let A_1 be the class of functionsf analytic in $U = \{z \in C; |z| < 1\}$, and normalized by f(0)=0, f'(0)=1 where C is the set of complex numbers. An f in A_1 with $f(z) \neq 0$ in the punctured disc $U/\{0\}$, may be expressed as $f(z) = \psi(g) = z/g(z)$ in U,

where $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ in U.

Mitrinovic [2], Reade et.al [3], Silverman and Silvia [6] and Srinivas [9] studied these coefficientsb$_n$'s.

Mitrinovic [2] obtained estimates for the radii on univalence of certain rational functions. In particular, he found sufficient conditions for functions of the form

(1) $\frac{z}{1+a_1+b_2z^h+\ldots+b_nz^n}$

$b_n \neq 0$, to be univalent in the unit disk U.

A function

(2) $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

in A_1 is said to be starlike of order, $0 \leq \alpha \leq 1$, if $\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha$ in U. The set of all such functions is denoted by $S^*(\alpha)$. The functions in $S^* \equiv S^*(0)$ are called starlike functions. Throughout this researchpaper we let f be of the form (2). A function f (z) in A_1 is said to be convex, if $\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$ in U. A function f(z) in A_1 is said to be convex of order $0 \leq \alpha \leq 1$, if $\operatorname{Re}\left\{1 + \frac{zf'' z}{f'(z)}\right\} > 0$ in U.

A continuous passage from starlike functions to convex functions is the following class of functions $\alpha -$ convex in U. A function fin A_1 is said to be α −convex in U, $\alpha \in C$, if $f(z)f'(z)/z \neq 0$ and β A generalization of this class is the family of function f in A_1 for which

(3) $\operatorname{Re}\left[\mu\frac{zf'}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0, z \in U.$

Let us denote this class by $CV(\lambda, \mu)$

A function $f \in A_1$ is said to be spirallike in U, if $\operatorname{Re}\left[\frac{1}{\cos\lambda}\left(e^{i\lambda}\frac{zf'(z)}{f(z)} - i\sin\lambda\right)\right] > 0$, in U. These functions generaliz starlike functions and were studied by Spacek[8]

Silvia [7] introduced the following generalization of $\alpha -$ convex and $\lambda -$ spirallike functions: A function f in A_1 is said to be $\alpha - \lambda$ spiral of order $\beta, a \geq 0, |\lambda| < \pi/2, \lambda$ real, $0 \leq \beta - 1$, if $f(z)f'(z)/z \neq 0$ for $z \in U$ and

$$\sec\lambda\left[\left(e^{i\lambda} - \alpha\right)\frac{zf'(z)}{f'(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] >$$

β.

Rebertson [4] generalized the concept of convex functions of order α as follows:

A function f in a_1 is said to be a $\lambda -$ Rebertson function of order in the unit disc U.

In the note [3], Reade et.al., showed that the Mitrinovic criterion for univalence of functions of the form (1) does not guarantee starlikeness and gave sufficient conditions for such functions to be (i) starlike of order α and (ii) convex, as $n \to \infty$

Functions $f \in A_1$ are said to be in the class $UCD(\alpha), \alpha \geq 0$, if

$\operatorname{Re} f'(z) \geq \alpha|zf''(z)|, z \in U$

In [10] Thomas et. al. studied $UCD(\alpha)$. In [5] the class T consisting of univalent functions f in A_1 of the form

(4) $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0,$

was investigated. By $TUCD(\alpha)$ we denote functions in $UCD(\alpha)$ of the form (4). For this class, the following necessary and sufficient condition was derived in Thomas et al [10] :

(5) $\sum_{k=2}^{\infty} k[1 + \alpha(k-1)a_k \leq 1.$

In this paper we derive conditions on b_n's necessary for f to be in $TUCD(\alpha)$ and sufficient for f to be in the class $CV(\lambda, \mu)$ in Sections 1 and 2 respectively. These sufficient conditions generalize some earlier results for some known classes.

SECTION – 1

Here we derive a necessary condition on the coefficients b_n's for the functions in $TUCD(\alpha)$:

Theorem 1 : If

$\psi(g) = \frac{z}{g(z)} = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ in U and $\psi(g) \in (g) \in TUCD(\alpha), 0 \le \alpha < 1$ then

(6) $0 \le b_n \le \frac{1}{(n+1)(1+\alpha n)}, n = 1,2,\ldots.$

The second inequality is sharp for

(7) $g_n(z) = 1 + \sum_{k=1}^{\infty} \left[\frac{1}{(n+1)(1+\alpha n)}\right]^k z^{nk}$

in U and

$\psi(g_n) = \frac{z}{g_n(z)} = z - \frac{1}{(n+1)(1+\alpha n)} z^{n+1}, z \in$ U.

Prof : Since $\psi(g) \in TUCD(\alpha)$ it has the Taylor series expansion

$\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \ge 0, z \in U$, By the definition of g(z),

(8) $b_n = \sum_{k=0}^{n-1} b_k a_{n-k+1}$

for $n \ge 1$ where $b_0 = 1$.

First we show that $\{b_n\}_1^{\infty}$ is a sequence of nonnegative real numbers. It follows from the equation (8) that $b_1 = a_2 \ge 0$. Now assume that $b_k \ge 0$ form $1 \le k \le n$, for some $n \in N$, the set of natural numbers. Since,

$b_{n+1} = \sum_{k=0}^{n} b_k a_{n+2-k}$

and a_k's are nonnegative, we have $b_{n+1} \ge 0$. This proves by induction that $b_{n+1} \ge 0$. This proves by induction that $\{b_n\}_1^{\infty}$ is a sequence of nonnegative real numbers.

By the necessary and sufficient condition (5) for f to be in $TUCD(\alpha)$:

(9) $\sum_{n=2}^{\infty} n[1+\alpha(n-1)]\, a_n \le 1.$

We have

$b_1 = a_2 \le \frac{1}{2(1+\alpha)}$

This proves the equality (6) for n = 1.

Now, let the inequality (6) be true for n, satisfying $1 \le n \le K$, for some $k \in N$. Then,

(10)

$b_{k+1} = \sum_{n=0}^{k} b_n a_{k+2-n} \le \sum_{n=0}^{k} \frac{1}{(n+1)(1+\alpha n)} a_{k+2-n}.$

Set, for $n \ge 2$,

$a_n = \lambda_n \frac{1}{n[1+\alpha(n-1)]}$

For $\psi(g) = z - \sum_{n=2}^{\infty} a_n z^n \in TUCD(\alpha)$ it is necessary, by (5) that

$\sum_{n=2}^{\infty} n[1+\alpha(n-1)]a_n \le 1$

Thus, $\lambda_2 \ge 0$ for $n \ge 2$ and

(11) $\sum_{n=1}^{k+1} \lambda_{n+1} \le 1$

The inequality (10) is equivalent to

$b_{k+1} \le \sum_{n=0}^{k} \lambda_{k+2-n} \frac{1}{(n+1)(1+n\alpha)}, \frac{1}{(k+2-n)[1+(k-n+1)\alpha]}$

(12) $\le \frac{1}{(k+2)[1+(k+1)\alpha]} \sum_{n=0}^{k} \lambda_{k+2-n}$

(13) $\le \frac{1}{(k+2)[1+(k+1)\alpha]}$

The inequality (12) holds since

$$(n+1)(1+n\alpha)(k+2-n)[1+(k-n+1)\alpha] \ge 2(k+2)[1+(k+1)\alpha]$$

$$\Leftrightarrow (1+\alpha n)(1+\alpha k-\alpha n)(k+1-n) + \alpha(k+1)(1+\alpha k) + \alpha^2(k+1) + \alpha[1+\alpha(k+1)]$$

$$\ge \alpha(n+1)(1+\alpha n)$$

which is true for $0 \ge n \le k$ and the inequality (13) holds due to (11). This proves the inequality (6) for $n = k+1$ and the proof of the theorem is complete by the induction argument. It is easily seen that sharpness of the second inequality in (6) is attained for the function $\psi(g_n)$ where g_n is as in the equation (7).

SECTION – 2

Next we determine a sufficient condition on f in terms of b_n's for the functional

$Re[\lambda(1 + zf''(z)/f'(z) + \mu zf'(z)/fz$

to be positive in the unit disc U so that such f is in $CV(\lambda,\mu)$

Theorem 2 Let $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n) \in A_1$ with $b_n's$ satisfying

(14) : -

$$\left((\lambda) + |2\lambda+\mu| + Re(\lambda+\mu)\,|b_1| + \sum_{n=2}^{\infty}[(\lambda) + |2\lambda+\mu| + Re(\lambda+\mu)\,(n-1)|b_n| \le Re(\lambda+\mu)]\right)$$

where λ, μ are in C and at least one of them is nonzero, Then $f \in CV(\lambda,\mu)$

Proof : For $f(z) = z/g(z)$ where $g(z) = (1+\sum_{n=1}^{\infty} b_n z^n), z \in U$, we have

(15) $\lambda\left(1+\frac{zf''(z)}{f'(z)}\right) + \mu\frac{zf'(z)}{f(z)} = \lambda + \mu\frac{(2\lambda+\mu)\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n} + \frac{\lambda\sum_{n=2}^{\infty} n(n-1)b_n z^n}{1+\sum_{n=2}^{\infty}(1-n)b_n z^n}$

in the unit disc U. For $\alpha = |2\lambda+\mu|\, Re(\lambda+\mu) / (|2\lambda+\mu| + |\lambda|)$, we have,

(16) $\left|\frac{(2\lambda+\mu)\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right| \le \frac{|2\lambda+\mu|\sum_{n=1}^{\infty} n|b_n|}{1-\sum_{n=1}^{\infty}|b_n|}$

and

(17) $\left|\frac{\lambda\sum_{n=2}^{\infty} n(n-1)b_n z^n}{1+\sum_{n=2}^{\infty}(1-n)b_n z^n}\right| \leq \frac{|\lambda|\sum_{n=2}^{\infty} n(n-1)|b_n|}{1-\sum_{n=2}^{\infty}(n-1)|b_n|} \leq Re(\lambda+\mu)-\alpha$

by the condition (14). By using the inequalities (16) and (17) in the equation (15), the inequality (3) in obtained, Hence $f \in CV(\lambda,\mu)$

Corollary 1 : If $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n)$ is in A_1 with the b_n' satisfying

$$(1+|\lambda|+|1+\lambda||b_1|) + \sum_{n=2}^{\infty}[1 + (1+|\lambda|+|1+\lambda|)n](n-1)|b_n| \leq 1$$

For $\lambda \in C$, then f is λ −convex in U.

Proof : By choosing $\mu = 1-\lambda$, (14) becomes the required sufficient condition.

Corollary 2 : (Reade et al. [3]) If $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n), z \in U$ with the b_n' satisfying

$$4|b_1| + \sum_{n=2}^{\infty}(n-1)(3n+1)|b_n| \leq 1$$

then f becomes convex in U.

Proof : By taking $\lambda = 1$ and $\mu = 0$ in Theorem 2, the corollary is obtained.

Corollary 3 : If $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n), z \in U$ with the $b_n's$ satisfying

$$(\cos\lambda + \alpha + |e^{i\lambda}+\alpha|)|b_1| + \sum_{n=2}^{\infty}[(\alpha+|e^{i\lambda}+\alpha|)n + \cos\lambda](n-1)|b_b| \leq \cos\lambda$$

For $|\lambda| < \pi/2, \lambda \in R$, the set of real numbers, $0 \leq \alpha$ then f is $\alpha-\lambda$ spiral of order o.

Proof : Choosing $\lambda = \alpha, e^{i\lambda}-\alpha$ in place of μ Theorem 2 gives the corollary.

Corolary4 : (Ahuja and Jain [1]) If $f(z) = z/(1+\sum_{n=1}^{\infty} b_n z^n), z \in U, -\pi/2 < \beta < \pi/2$ and

$$(3+\cos\beta)|b_1| + \sum_{n=2}^{\infty}(3n+\cos\beta)(n-1)|b_n| \leq \cos\beta$$

then f is a β −Robertson function of order 0 in U.

Proof : For $\mu = 0, \lambda = e^{i\beta}, -\pi/2 < \beta < \pi/2$. Theorem 2 gives the corollary.

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