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Topological Properties of Some Hyperspaces of Convex Bodies Associated with Riemannian Manifold

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ABSTRACT: In this paper, we study the topological properties of some Gromov – Hausdorff hyperspaces on the set of isometry classes of positively curved C^∞ convex bodies in $\mathbb{R}^n$ and determine their homeomorphismtype. We prove that some of these hyperspaces are homogeneous and state where homogeneity failed on them. This is done by matching the tools of convex geometry with what is required by the infinite dimensional topology.

Key words: Topological spaces, Riemannianmanifold, Hyperspace, Homeomorphism, ANR, SDAP, Polish space, Homogeneous space, convex set and convex body.

I. INTRODUCTION

The Gromov – Hausdorff distance is a useful tool for studying topological properties of families of Riemannian metrics. For two compact metric spaces X and Y, the number $d_{GH}(X, Y)$ is defined as the infimum of all Hausdorff distances $d_H(i(X), j(Y))$, for all metric spaces M and all isometric embeddings $i: X \to M$ and $j: Y \to M$. [Antonyan, 2016]

Clearly, the Gromov – Hausdorff distance between isometric spaces is zero; it is a metric on the familyof isometry classes of compact metric spaces. The metric space (GH, d_{GH}) is called the Gromov – Hausdorff Hyperspace. When d_{GH} is well understood, we simply write GH as the Gromov– Hausdorff hyperspace.

A metric is intrinsic if the distance between any two points is the infimum of the length of curves joining the points. Any C^∞ Riemannian metric is intrinsic and this property is preserved under Gromov – Hausdorff limit. By [Jiwon, 2019], for $k \in \mathbb{R}$, let ${}_{\text{curv}}{}^{GH}_{\geq k}(M)$ be the Gromov – Hausdorff hyperspaces of intrinsic metric of curvature $\geq k$ on M. Let ${}_{\text{sec}}{}^{GH}_{\geq k}(M)$, ${}_{\text{sec}}{}^{GH}_{> k}(M)$ be the Gromov – Hausdorff hyperspaces of C^∞Reimannian metric on M of sectional curvatures $\geq k$, $> k$ respectively. Topological properties of these hyperspaces are largely a mystery which is why it is more common to give ${}_{\text{sec}}{}^{GH}_{> k}(M)$, the C^∞ topology resulting in a stratified space whose strata are Hilbert manifolds. In this paper, we study topological properties of some Gromov – Hausdorff hyperspaces on the set of isometry classes of positively curved C^∞ convex bodies in $\mathbb{R}^n$ and determine their homeomorphism type. We prove that some of these hyperspaces are homogeneous and state where homogeneity failed on some of those hyperspaces. In this paper, we are going to make use of some notions of infinite dimensional topology and convex geometry as in [Luisa, etal., 2021] to obtain our own results. Throughout this paper, ω represents the set of nonnegative integers.

Also,We consider the following notations for hyperspaces of Euclidean space$\mathbb{R}^n$. $\mathcal{K} = \{\text{convex compacta in } \mathbb{R}^n\}$

$\mathcal{K}_s = \left\{\begin{matrix}\text{convex compacta in } \mathbb{R}^n \text{ with} \\ \text{steiner point at the origin}\end{matrix}\right\}$ and $\mathcal{K}_s{}^{k\trianglelefteq} = \mathcal{K}^{k\trianglelefteq} \cap \mathcal{K}_s$. $B_p = \{\text{convex bodies } D \in \mathcal{K}s \text{ with intrinsically } C\infty \text{ boundary of sec} > 0$,

$B_d = \{\text{convex bodies } D \in \mathcal{K}_s \text{ with intrisically } C^\infty \text{ boundary metrics}\}$,

$B^{k,\propto} = \{C^{k,\propto} \text{ convex bodies in } \mathcal{K}_s\}$, $B^k = B^{k,0}$ if it is continuous. If $0 < \alpha < 1$, we say it is α −Holder continuous and if $\alpha = 1$, we say it is Lipschitz kth partial derivative function.

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We begin by letting $M =$ boundary sphere $\mathbb{S}^{n-1}$ and $k = o$, the above $\mathcal{GH}$ hyperspace can be identified with the $O(n) -$ quotients of certain hyperspace of $\mathbb{R}^n$, see **Lemma 1.1** below

Lemma 1.1[Belegradek, 2017]**:** The map $\mathcal{K}_s{}^{n\leq n-1}/O(n) \to {}_{\sec\geq 0}^{\mathcal{GH}}(\mathbb{S}^{n-1})$ that assigns to the congruence class of a convex compactum, the isometry class of its boundary surface is a homeomorphism which restricts to homeomorphisms $B_d/O(n) \to {}_{\sec\geq 0}^{\mathcal{GH}}(\mathbb{S}^{n-1})$ and $B_p/O(n) \to {}_{\sec>0}^{\mathcal{GH}}(\mathbb{S}^{n-1})$. So, the boundary surface of a $n-1$ dimensional convex compactum K is the double of K along the boundary with the subspace intrinsic metric, as in [Valov, 2020].

Consider the Hilbert cube $Q = [-1,1]^{\omega}$ and its radial interior $\Sigma = \{(t_i)_{i\in\omega} \text{ in } Q: \sup_{i\in\omega}|t_i| < 1\}$.Where ω represents the set of nonnegative integers and the superscript ω is the countably many copies of the space.

Also we heavily make use of the map $s: \mathcal{K}(\mathbb{R}^n) \to C(\mathbb{S}^{n-1})$, given by $s(D) = h_D \backslash \mathbb{S}^{n-1}$, which enjoys the following properties, as in [Belegradek, 2017].

- s is an isometry onto its image, where as usual the domain has the Hausdorff metric and codomain has the metric induced by the C^0 norm.
- The image of s is closed and convex.
- s is Minkowski linear, i.e, $s(aD + bK) = as(D) + bs(K)$, for any nonnegative a, b and $D, K \in \mathcal{K}$.

The steiner point is a map $s: \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}^n$, given by

$$s(D) = \frac{1}{\mathrm{vol}\,(B^n)} \int_{\mathbb{S}^{n-1}} u h_D(u) du,$$

which has the following properties;

(i) s is Lipschitz

(ii) s is invariant under rigid motions, $s(gD) = gs(D)$ for any $g \in \mathrm{Iso}(\mathbb{R}^n)$

(iii) s is Minkowski linear, i.e, $s(aD + bK) = as(D) + bs(K)$, for any nonnegative a, b and $D, K \in \mathcal{K}(\mathbb{R}^n)$,

(iv) $s(D)$ lies in the relative interior of D

(v) D is a point such that $s(D) = D$ (D is fixed point) and

(vi) s is the only continuous Minkowski linear, $\mathrm{Iso}(\mathbb{R}^n)$ – invariant map from $\mathcal{K}(\mathbb{R}^n)$ to $\mathbb{R}^n$.

Thus, the hyperspace $\mathcal{K}_s$ of convex compacta in $\mathbb{R}^n$ with steiner point at o is an $O(n)$ –invariant closed convex subset of $\mathcal{K}$ and the map $\mathcal{K} \to \mathbb{R}^n \times \mathcal{K}_s$ sending D to $(s(D), D - s(D))$ is homeomorphism. [Schneider, 2020]

II. DEFINITION OF SOME RELEVANT TERMS

Definition 2.1 [Burago, etal, 2001]**:** A manifold M is said to be Reimannian manifold if the Reimannian metric is defined on it.

Definition 2.2 [Osipov& Oscar, 2017]**:** A subspace $A \subset X$ of topological space X is a subset of X with subspace (Induced) topology.

Definition 2.3 [Burago, etal., 2001]**:** Given that X and Y are topological spaces. Then Y is X-manifold if each point of Y has a neighborhood homeomorphic to an open subset $O \in X$.

Definition 2.4 [Fernandez &Unzueta , 2018 **]:** A subspace $A \subset X$ is homotopy dense if there exist a homotopy $h: X \times I \to X$ with $h_o = id$ and $h(X \times 0,1 \subset A$. If X is an ANR, then $A \subset X$ is homotopy dense iff each map $I^k \to X$ with $k \in \omega$ and $\partial I^k \subset B$, can be uniformly approximated real boundary by map $I^k \to B$, where B is a closed subset of X.

Definition 2.5 [Ahmadu&Morawo, 2021]: A closed subset $B \subset X$ is $Z -$ set, if every map $f: B \to X$ can be uniformly approximated by a map whose range misses B.

Definition 2.6 [Valov,2020]**:** A $\sigma Z -$ set is a countable union of $Z -$ set.

Definition 2.7 [Valov, 2020]**:** An embedding is a $Z -$ embedding if its image is a $Z -$ set.

Definition 2.8 [Valov. V, 2020]**:** A topological space X is said to be $\sigma -$ compact, if it is countable union of compact sets.

Definition 2.9 [Jiwon, 2019**]:** A function $f: X \to Y$ between topological spaces X and Y is a homeomorphism if f is a bijection and continuous with continuous inverse.

Definition 2.10 [Facundo, 2012]**:** A hyperspace of $\mathbb{R}^n$ is a set of compacta of $\mathbb{R}^n$ equipped with the Hausdorff metric.

Definition 2.11 [Antonyan, 2016]**:** A topological space X is a Polish space if it admits a complete metric.

Definition 2.12 [Higueras& Montano, 2020]**:** A convex body is a convex set with nonempty interior.

Definition 2.13 [Higueras& Montano, 2020]**:** Let $\mathcal{A}$ be a class of spaces $\mathcal{M}_0$ or $\mathcal{M}_2$. A space X is $\mathcal{A} -$ universal if each space in $\mathcal{A}$ is homeomorphic to a closed subset of X.

Definition 2.14 [Antonyan, 2016]**:** A topological space X is said to be strongly $\mathcal{A} - universal$ if for every open cover $\mathcal{U}$ $of\ X, each\ A \in \mathcal{A}$, every closed subset $A_1 \subset A$, and each map $f: A_1 \to X$ that restrict to a $Z - embedding\ on\ A_1$, there exist a $Z - embedding\ f_1: A_1 \to X\ with\ f_1 \backslash A_1 = f \backslash A_1$ such that $f, f_1\ are\ \mathcal{U} - closed$.

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Definition 2.15[Belegradek, 2017]: The Steiner point is a way to assign a centre to any convex compactum in $\mathbb{R}^n$ that is continuous, Iso($\mathbb{R}^n$) – invariant , Minkowski linear and fixed point, these properties characterizes the Steiner point.

III. THE RESULTS OBTAINED

The new ingredient, stated in the **theorem 3.1** below, follows from a version of Cheeger – Gromov compactness theorem, by restate the Cheeger – Gromov compactness theorem as follow; Associated to the class of smooth and compact Riemannian n – manifold, there is finite list of smooth manifolds $(M_1, M_2, \dots M_l)$, with the following properties;

For any M in class of smooth and compact Riemannian n- manifold, there is a diffeomorphism $f: N_k \to M, 1 \leq k \leq l,$ and an atlas charts covering N_k under which f^*g is uniformly controlled in $C^{1,\propto}$ in term of constant that determine the class, as in [Knox, 2013] and [Anderson, 2021].

Now, for $k \in \mathbb{R}$, Let $\mathcal{GH}_{sec \geq k}(M)$ *and* $\mathcal{GH}_{sec > k}(M)$ be the GH hyperspace of C^∞ Riemannian metrics on M with sectional curvatures $\geq k, > k$, respectively. By combining method of [Anderson, 2021] and [Belegradek, 2017], we have **theorem 3.1** below by letting Riemannian manifold $M = boundary\ sphere\ \mathbb{S}^{n-1}\ and\ k = 0$, then we identified Gromov – Hausdorff hyperspace of $boundary\ sphere\ \mathbb{S}^{n-1}$ with the $O(n) - quotients$ of certain hyperspace of $\mathbb{R}^n$ as we have stated in the methodology.

Theorem3.1.: Given that $\mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1})$ and $\mathcal{GH}_{sec > 0}(\mathbb{S}^{n-1})$ are GH hyperspaces of Riemannian metrics with sectional curvatures $\geq 0, > 0$. Then , the space $\mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1}) \backslash \mathcal{GH}_{sec > 0}(\mathbb{S}^{n-1})$ is F_σ subset of $\mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1})$ and a countable intersection of $\sigma - compact$ sets.

Proof: For $k, l \in \mathbb{Z}$ such that $k \geq n-1$ and $l \geq n-2$. Let $Q_l^k \subset \mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1})$ be the subset consisting of the isometry classes of metrics whose sectional curvature vanishes somewhere, the diameter of this metric is in the closed and bounded interval $0 \leq x \leq l$, so that the injectivity of its radius is at least $\frac{1}{l}$ and the C^0 norms of the curvature tensor and every covariant derivative of the curvature tensor of order $1, \dots, k$ is at most l. Then $cl(Q_l^k) \in \mathcal{GH}_{sec > 0}(\mathbb{S}^{n-1})$ is compact and $cl(Q_l^k) \cap \mathcal{GH}_{sec > 0}(\mathbb{S}^{n-1}) = \emptyset$, because for each $\alpha \in (0,1)$, there exist a sequence $x_n \in Q_l^k$ that subconverges in the $C^{k,\alpha}$ topology to an isometry class of a $C^{k+1,\alpha}$ Riemannian manifolds. Again, since $k \geq n-1$, the sectional curvature will certainly vanish in the limit of the sequence $x_n \in Q_l^k$. Hence, for each k, we have $\mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1}) \backslash \mathcal{GH}_{sec > 0}(\mathbb{S}^{n-1}) = \bigcup_{l \geq n} cl(Q_l^k) \cap \mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1})$ which defines $\mathrm{F}_\sigma \in \mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1})$ satisfies the following $(i) Iso(C^{k+1}) \in \bigcup_{l \geq n} cl(Q_l^k)$, $(ii) \mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1}) \backslash \mathcal{GH}_{sec > 0}(\mathbb{S}^{n-1}) \subset \bigcup_{l \geq n} cl(Q_l^k)$ and $\bigcup_{l \geq n} cl(Q_l^k) \cap \mathcal{GH}_{sec > 0}(\mathbb{S}^{n-1}) = \emptyset$, Therefore, $\mathcal{GH}_{sec \geq 0}(\mathbb{S}^{n-1}) \backslash \mathcal{GH}_{sec > 0}(\mathbb{S}^{n-1}) = \bigcap_{k \geq n-1,} \bigcup_{l \geq n-2} cl(Q_l^k)$. ■

The next results is a sequel to [Belegradek, 2017] and [Valov, 2020], where the author used convex geometry and infinite dimensional topology to determine the homeomorphism type of some convex compacta in $\mathbb{R}^n$, some convex bodies and also derive a number of properties of their $O(3) -$ quotients. In particular, in [Belegradek, 2017], the author isolated some conditions on a hyperspace $\mathfrak{D}$ with $B_p \subset B_d \subset B^{1,1}$ that give the conclution of **theorem 3.2** below with B_d replaced by $\mathfrak{D}$ so that the condition hold, e.g, if D/B_p is $\sigma - compact$,which includes the case $\mathfrak{D} = B_p$. Here, we verify the conditions for $\mathfrak{D} = B_d$

Lemma 3.1 [Belegradek, 2017]**:** If $B_d \subset X \subset \mathcal{K}_s$, then X is an AR that is homotopy dense in $\mathcal{K}_s$.

Theorem 3.2: Suppose $A \subset Q^\omega \backslash \Sigma^\omega$ is a $Z - set$ homeomorphic to suspension of real projective plane SRP^{n-1}. Then, (i) there exist a unique homeomorphism $h: \mathcal{K}^{n-1 \leq n} \to Q^\omega \backslash A\ such\ that\ h(B_d) = h(B_p) = \Sigma^\omega$, and (ii) $\mathcal{K}^{n-1 \leq n}$ is a contractible Q – manifold which is obtain from Q by deleting a $Z - set\ A$ such that there exist a homeomorphism $f: A \to SRP^{n-1}$.

Proof: (i)We begin the proof by showing that between two hyperspaces $B_d\ and\ \Sigma^\omega$, there is a homeomorphism $f: B_d \to \Sigma^\omega$. Due to the fact that for every convex body $\mathfrak{D} \in B_d$, there is $C^{1,1}$ boundary that is C^∞ at point of intrinsically positive curvature so that $B_p \subset B_d \subset B^{1,1}$. So, we have that B_d is an AR , SDAP and σZ.

The $O(n) - orbit$ map from $B_p\ and\ B_d$ onto the set of congruence classes are continuous and proper. By taking the preimage of a continuous proper map that preserves being $\mathrm{F}_{\boldsymbol{\sigma}}\ and\ \sigma - compact$, then the preimages$B_d \backslash B_p$ is $\mathrm{F}_{\boldsymbol{\sigma}} \in B_d$ and hence is a countable intersection of $\sigma - compact\ sets$, this implies that $B_d \in \mathcal{M}_2$and strongly $\mathcal{M}_2 - universal$. Therefore, there is a homeomorphism $f: B_d \to \Sigma^\omega$.

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Next, the pair $(\mathcal{K}^{n-1\leq n}, B_d)$ is $(\mathcal{M}_o, \mathcal{M}_2) - arbsorbing$ and $\mathcal{K}^{n-1\leq n}$ is homeomorphic to the complement $C \subset Q^\omega\ of\ a\ Z-set$ that is homeomorphic to the suspension $SRP^{n-1}\ over\ RP^{n-1}$. Provided that the hyperspace Σ^ω is convex , $cl(\Sigma^\omega) = Q^\omega$ and homotopy dense in Q^ω. Thus, every compact subset $K \subset Q^\omega \backslash \Sigma^\omega\ is\ a\ Z-set.$ Further, provided that $A \subset Q^\omega \backslash \Sigma^\omega$ as written in the statement of the theorem, there exist an homeomorphism $h: Q^\omega \backslash A \to \mathcal{K}^{n-1\leq n}\ of\ Q^\omega\ which\ maps\ Q^\omega \backslash A\ to\ \mathcal{K}^{n-1\leq n}$. Then the pair $(Q^\omega \backslash A, \Sigma^\omega)\ is\ (\mathcal{M}_o, \mathcal{M}_2) - arbsorbing.$ The uniqueness of the absorbing pair proved the first statement.

(ii) Let $\propto k$ be a one – point compactification of k, provided that the inclusion $Q \subset \propto k$ is an open map, then there is homeomorphism $h: \mathcal{K}^{n-1\leq n} \to Q\ such\ that\ h: \propto k \to Q.$ Since any point of $Q\ is\ a\ Z-set$, then $\propto k$ map any $Z-set\ X \in K$ to a $Z-set\ Y \in Q$. Then $\propto k^{n-1\leq n}$ is a $Z-set\ in\ Q\ such\ that\ h: Q \to SRP^{n-1}.$ Therefore, there is homeomorphism $h: \mathcal{K}^{n-1\leq n} \to SRP^{n-1}$. ■

In [Belegradek, 2017], one finds a hyperspace $\mathfrak{D}$ with $B_p \subset B_d \subset B^{1,1}$ such that $\mathfrak{D} \backslash B_p$ embeds into cantor set , B_p is open in $\mathfrak{D}$ and $\mathfrak{D}$ is not topologically homogeneous, and in particular, not homeomorphic to Σ^ω. We explain the example of [Belegradek, 2017] further in **theorem 3.3 and 3.4** below by trying to isolate the condition on a hyperspace that would make it homeomorphic to Σ^ω by the help of **lemma 3.2** below, we fix $\alpha \in I$ and let $\mathfrak{D}$ denote the arbitrary hyperspace of Riemannian manifold satisfying $B_p \subset \mathfrak{D} \subset B_d \subset B^{2,\alpha}$.

Lemma 4.2: [Belegradek, 2017]**:** If $\mathfrak{D}$ is an arbitrary hyperspace of $\mathbb{R}^n$ such that $B_d \subset \mathfrak{D} \subset B^{2,\alpha}$. Then $\mathfrak{D}$ has SDAP.

Theorem 3.3.: Any topological space is homeomorphic to a subset $O \subset B^{2,\alpha} \backslash B_d$ such that $B_d \subset O \cup B_d$ is open.

Proof: Let $A \in B^{2,\alpha}$ whose support functionh_A is not is not smooth manifoldC^∞. For $t \in I$, and $t \neq 0$, consider the map $f_t: B_d \to B^{2,\alpha}$ given by $f_t(D) = tA + (1-t)D$, so that the image $Y\ of\ f_t$ belong to $B^{2,\alpha}$ but $Y \notin B_d$ when t≠ 0, due to the fact that if $h_f(D) \in B_d$, then h_A is a linear combination of C^∞ functions. For each t≠ 0, the map f_t is a topological embedding. (Certainly, the map is one - one as we can eliminate tA and moreover, for each D_1 and D_2, if $f_t(D_1)$, $f_t(D_2)$ are closed , then their support functions are also closed, and after eliminating tA, the support functions of $(1-t)D_1, (1-t)D_2$are closed, for this, D_1 and D_2 are also closed). Now, provided that the convex body B_d is homeomorphic to Σ^ω, then the space $B^{2,\alpha} \backslash B_d \supset C$, where C is a topological copy of Q and $cl(C) \in (Q \cup B_d)$, since the Hilbert manifold Q is both closed and bounded. Any separable metric space embeds into Q. If Y is the image of such an embedding into the above copy Q , then, Y is closed in $Y \cup B_d$ ■

Theorem 3.4: Given that for $\alpha > 0$, there is arbitrary hyperspace $\mathfrak{D}$ such that $B_p \subset \mathfrak{D} \subset B_d \subset B^{2,\alpha}$. Then if (i) $\mathfrak{D} \backslash B_p$ is $\sigma - compact,$ both $\mathfrak{D}\ and\ B_p$ are homeomorphic to Σ^ω (ii) $\mathfrak{D} \backslash B_p$ embeds into the cantor set such that $B_p \subset \mathfrak{D}$ is open, then $\mathfrak{D}$ is not topologically homogeneous, and not a $\Sigma^\omega - manifold.$

Proof: (i) Since we know that any $\sigma - compact\ subspace\ A\ is\ F_\sigma$and $A \subset \mathcal{M}_2$**.** By **lemma 3.2** and due to the fact that if the hyperspace B_p is $G_\delta \in \mathfrak{D}$, then $\mathfrak{D}$ is strongly $\mathcal{M}_2 - universal, \mathfrak{D}\ is\ AR\ with\ SDAP, \sigma Z,$ strongly $\mathcal{M}_2 - universal\ and\ \mathfrak{D} \in \mathcal{M}_2$. Therefore $\mathfrak{D}$ is $\mathcal{M}_2 - absorbing$ and only space Σ^ω bear such character.

(ii) For any uncountable polish space X such as cantor set, the Borel hierarchy of its subsets does not stabilize, so in particular, it countains a subset not in $\mathcal{M}_2$. We can use **theorem 3.3** above to embeds it onto a subset $O \subset B^{2,\alpha} \backslash B_p\ such\ that\ cl(B^{2,\alpha} \backslash B_p) \subset O \cup B_p$.

Since $\mathcal{M}_2$ is closed hereditary, $O \cup B_p \notin \mathcal{M}_2$ and hence not a $\Sigma^\omega - manifold.$ If $O \cup B_p$ were topologically homogeneous, then it will be a $\Sigma^\omega - manifold$ because the $\Sigma^\omega - manifold B_p$ is open in $O \cup B_p$.Therefore, $\mathfrak{D}$ is not topologically homogeneous. ■

In the next result, we used method of [Arhangel'skill&Vann Mill, 2020] and [Banakh, 2021] as in **lemma 3.3 – 3.5** below,to explain the hyperspaces $B_d$and $\mathfrak{D}$ further, we state condition for which B_d and $\mathfrak{D}$ are strongly $\mathcal{M}_2 -$ universal and topologically homogeneous.

Lemma 3.3 [Banakh, 2021]: Let C be a class of spaces , M be an ANR and X be a homotopy dense subset of M such that X has SDAP. If for some pair (K, C), the pair (M, X) is strongly (K, C) – universal, then the space X is strongly $C -$ universal.

Lemma 3.4 [Belegradek, 2017]**:**B_d is strongly $\mathcal{M}_2 -$ universal.

Lemma 3.5 [Banakh, 2021]**:** $\mathfrak{D}$ has SDAP

Theorem 3.5: SupposeB_d is hyperspace of convex bodies with intrinsically C^∞ boundary metrics and $\mathfrak{D}$ be an arbitrary hyperspace such that B_d is G_δ in

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$\mathfrak{D}$. Then (i) $B_d \in \mathcal{M}_2$ and is topologically homogeneous (ii) $\mathfrak{D}$ is strongly $\mathcal{M}_2 -$ universal

Proof: Let B_d^∞ denote the set B_d with smoothC^∞ topology. Here, the Gaussian curvature of any convex body $\mathfrak{D} \in B_d^\infty$ varies continuously. Thus, the set B_d^∞ is precisely the subset of hypersurfaces of positive Gaussian curvature in the space of all compact C^∞ hypersurfaces in $\mathbb{R}^n$ equipped with the C^∞ topology.The later space is Polish. Any open subset of a Police space is Police, hence the set B_d^∞ is Polish space.

Then, for $\rho \in \{0, \infty\}$, let $\Gamma^\rho(S^{n-1})$ denote the set $C^\infty(S^{n-1})$ equipped with the C^ρ topology. Let $S^\infty: B_d^\infty \to \Gamma^\infty(S^{n-1})$ be the map that associates to a convex body its support function, i.e, $S^\infty = S$ as maps of sets. Similarly to S, the map S^∞ is a topological embedding because the support function for sets in B_d equals the distance to O from the support hyperplane, and both the tangent plane and the distance to O vary in the C^∞ topology as O lies in the interior of each set in B_d. Since B_d^∞ is Polish space , its homeomorphic S^∞ - image is G_δ, so that the identity map $\Gamma^\infty(S^{n-1}) \to \Gamma^\rho(S^{n-1})$ takes any G_δ subset to a space in $\mathcal{M}_2$. Thus, $B_d \in \mathcal{M}_2$.

Next, since the hyperspace

$B_d = \{\text{convex bodies } D \in \mathcal{K}_s \text{ with intrisically } C^\infty \text{ boundary metrics}\}$, it shows that the map $S^\infty: B_d^\infty \to \Gamma^\infty(S^{n-1})$ is fixed point. Therefore, B_d is topologically homogeneous.

Also, by lemma**lemma 3.3** above , an ANR with SDAP is strongly $\mathcal{M}_2 -$ universal if and only if it contains a strongly $\mathcal{M}_2 -$ universal homotopy dense G_δ subset. By assumption, B_d is G_δ in $\mathfrak{D}$. Then the space B_d is strongly $\mathcal{M}_2 -$ universal, due to **lemma 3.4** above. By **lemma 3.1**, the hyperspace $\mathfrak{D}$ is an ANR with SDAP and B_d is homotopy dense in X. Hence $\mathfrak{D}$ is strongly $\mathcal{M}_2 -$ universal. ■

IV. REMARKS ON THE RESULTS OBTAINED

Some topological properties such as homeomorphism, AR, $\sigma -$ compactness on the hyperspaces $\mathfrak{D}, B_d$ and B_p which are the main objects of this work have been studied in [Belegradek, 2017] and [Banakh, 2021] but their homogeneity, ANR, SDAP, Polish, isometric embedding and strongly $\mathcal{M}_2 -$ universal have not been discussed in those papers. This presentation explained further, the topological properties that have already been discussed on the following hyperspaces $\mathfrak{D}, B_d$ and B_p, and defined the homogeneity, ANR, Polish, strongly $\mathcal{M}_2 -$ universal that have not been discussed before.

V. CONCLUSION

Topological properties of hyperspaces of Riemannian manifold have been studied and the homeomorphism types of the following hyperspaces of Euclidean space $\mathbb{R}^n$ such as convex compacta, convex bodies and strictly convex bodyare known.

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