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η-Ricci Solitons on f-Kenmotsu Manifold

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ABSTRACT: η- Ricci solitons on f-kenmotsu manifold are considered an manifolds satisfying certain curvature conditions: R(ξ ,X).S=0, S(ξ,X).R=0, $W_2(\xi, X).S=0$, and $S(\xi, X).W_2=0$. We proved that in f-Kenmotsu manifold (M, ϕ, ξ, η, g).Then the existence of an η-Ricci solitons implies that M is Einstein manifold and if the Ricci curvature tensor satisfies, S(ξ,X).R=0, then Ricci solitons M is steady. If the condition μ=0, then λ=0, which shows that λ is steady.The para-contact from
η-closed and the Nejnhuis tensor of the structural endomorphism ϕ is identically vanishes.

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Key words: Ricci and η -Ricci solitons, f - Kenmotsu manifolds, η -Ricci solitons, W_2 curvature tensor etc.

I. INTRODUCTION:

The concept of Ricci solitons was introduced by Hamilton [21] in the Year 1982. In differential geometry, a Ricci solitons is a special type of Riemannian metric such metric evalve under Ricci flow only be symmetries of the flow and ofter arise as limits of dilations of Singularities in the Ricci flow [22, 10, 14, 34]. They can be viewed as generalizations of Einstein metrics. They can viewed as fixed point of the Ricci flow, as a dynamical system on the space of Riemmanian metrics moduls diffeomorphisms ans scallings. Ricci solitons have been studied in many contexts: on Kahler manifolds [16], on contact and Lorentzian manifolds [1, 11, 2, 33, 35], on Sasakian [4, 20, 23], α-Sasakian manifolds [2] and K-contact manifolds [33], on Kenmotsu [3, 24, 26, 27] and f-Kenmotsu manifolds [11] etc. In Para contact geometry, Ricci solitons firstly appeared in the papers of G. Calvaruso and D. Perrone [13], Recently, C. L. Bejan and M. Crasmareanu studied Ricci solitons on 3-dimensional normal Para contact manifolds [35]. As a generalizations of Ricci solitons, the notion of η -Ricci solitons introduced by J. T. Cho and M. Kimura [15] which was treated by C. Calin and Crasmareanu on Hopf Hypersurfaces in complex space forms [12]. The concept is named after Gegorio Ricci-Curbastro. In 2015, A.M. Blaga have obtained some results on η-Ricci solitons satisfying certain curvature conditions.

In 1982, Hamilton [21] introduced the notion of Ricci flow to find a Canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman [31] used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}. \tag{1.1}$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci solution if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci solitons (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that

$$\mathcal{L}_V g + 2S+2\lambda g = 0, \tag{1.2}$$

where S is a Ricci tensor, $\mathcal{L}_V$ is Lie-derivative operator along the vector field V on M and λ is a real number.The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman [31], applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In 2008, Sharma [31], studied the Ricci solitons in contact geometry [35]. There after Ricci solitons in contact metric manifolds have been studied by various authors such as Bejan and Crasmareanu [12], Blaga [8, 9], Bejancu and Duggal [6], Bejan and Crasmareanu [5, 7] Nagaraja and Premalatta [26, 27] and many others.

II. PRELIMINARIES:

Let M be a real (2n+1)-dimensional differentiable manifold endowed with an almost contact structure (ϕ, ξ, η, g) satisfies

$$\phi^2=-I+\eta\otimes\xi,\ \eta(\xi)=1, \quad (2.1)$$

$$\phi.\xi=0,\ \eta o\phi=0,\ \eta(X)=g(X,\xi), \quad (2.2)$$

$$g(\phi X, \phi Y)=g(X,Y)-\eta(X)\eta(Y), \quad (2.3)$$

for any vector field $X,Y\epsilon\chi(M)$,where I is identity of the tangent bundle TM, ϕ is a tensor field and g is a metric tensor field, we say that (M,ϕ,ξ,η,g) is a f-Kenmotsu manifold if the Levi-Civita connection of g satisfy

$$(\nabla_X\phi)(Y)=f[g(\phi X,Y)-\eta(Y)\phi X], \quad (2.4)$$

where $f\epsilon c^{\infty}(M)$, such that $df\wedge\eta=0$, if

$$f=\alpha=\text{constant}\neq 0. \quad (2.5)$$

Then the manifold is α-Kenmotsu manifold. 1-Kenmotsu manifold is Kenmotsu manifold, if f=0, then the manifold is cosymplectic, an f-Kenmotsu manifold is said to be regular if $f^2+f'\neq 0$, where $f'=\xi(f)$. For an f-Kenmotsu manifold from (2.5) if follows that

$$\nabla_X\xi=f[X-\eta(X)\xi], \quad (2.6)$$

which on using equation (2.7), we have

$$(\nabla_X\eta)(Y)=f[g(X,Y)-\eta(X)\eta(Y)], \quad (2.7)$$

The condition $df\wedge\eta=0$, holds if dim $M\geq 5$ this does not hold in general if dim M=3 as is well known a 3-dimensional Riemannian manifold, we always have

$$R(X,Y)Z=g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y-\frac{\tau}{2}[g(Y,Z)X-g(X,Z)Y], \quad (2.8)$$

In a 3-dimensional manifold M, we have

$$R(X,Y)Z=(\frac{\tau}{2}+2f^2+f')[-g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y], \quad (2.9)$$

$$S(X,Y)=(\frac{\tau}{2}+f^2+f')g(X,Y)-(\frac{\tau}{2}+3f^2+3f')\eta(X)\eta(Y), \quad (2.10)$$

$$QX=(\frac{\tau}{2}+f^2+f')X-(\frac{\tau}{2}+3f^2+3f')\eta(X)\xi, \quad (2.11)$$

where R denotes the curvature tensor S is the Ricci tensor, Q is the Ricci operator and τ is the scalar curvature tensor on M. Taking from above equation (2.10), we obtain

$$R(X,Y)\xi=-(f^2+f')[\eta(Y)X-\eta(X)Y], \quad (2.12)$$

and (2.10) yields

$$S(X,\xi)=-2(f^2+f')\eta(X), \quad (2.13)$$

for any vector field X,Y,Z on M and R is the Riemannian curvature tensor S is the Ricci tensor of type (0,2). If the Ricci tensor of an almost contact Riemannian manifold M is of the form

$$S=ag+b\ \eta\otimes\eta, \quad (2.14)$$

for some function a and b on M, then M is said to be an η-Einstein manifold.

We now state and prove some basic results in a f-Kenmotsu manifold which will be frequently used later on. In Pokhariyal and Mishra have defined the curvature tensor W_2 given by [25,32]

$$W_2(X,Y,U,V)=R(X,Y,U,V)+\frac{1}{n-1}[g(X,U)S(Y,V)-g(Y,U)S(X,V)], \quad (2.15)$$

where S is a Ricci tensor of type (0,2).

Consider an f-Kenmotsu manifold satisfying W_2=0,(2.16), then we have

$$R(X,Y,U,V)=\frac{1}{n-1}[g(X,U)S(Y,V)-g(Y,U)S(X,V)], \quad (2.16)$$

Putting Y=V= ξ, in (2.16), then using equation (2.13), we obtain

$$S(X,U)=-\frac{2(f^2+f')}{(n-1)}[-g(X,Y)+\eta(U)\eta(X)], \quad (2.17)$$

Thus M is an Einstein manifold.

Theorem (2.1): If η-Ricci solitons on f-Kenmotsu manifold M, the condition W_2=0, holds, then M is an Einstein manifold.

Definition: η-Ricci solitons on f-Kenmotsu manifold is called W_2-Semi-Symmetric if it satisfies

$$R(X,Y).W_2=0,$$

where R(X,Y) is to be consider as a derivation of tensor algebra at each point of the manifold for tangent vector X and Y. In an f-Kenmotsu manifold the W_2-curvature tensor satisfies the condition

$$\eta(W_2(X,Y)Z)=0, \quad (2.18)$$

III. η-RICCI SOLITONS ON F-KENMOTSU MANIFOLD.

Let (M,ϕ,ξ,η,g) be a η-Ricci solitons on f-Kenmotsu manifold.

Consider from the above equation

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta\otimes\eta =0, \quad (2.19)$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is Ricci tensor field of the metric g, λ and μ are real constants. Writing $L_\xi g$ in terms of the Lie derivative ∇, we obtain

By virtue of the equations (3.1) and (2.4) in above equation, we get

$$2S(X,Y)=- g(\nabla_X\xi, Y)- g(X,\nabla_Y\xi)-2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y), \quad (3.2)$$

In view of the equations (3.2) and (2.4), in above equation, we get

$$S(X,Y)=(-f+\lambda)\, g(X,Y)+(f+\mu)\,\eta(X)\eta(Y), \quad (3.3)$$

for any X,Y $\epsilon\chi(M)$, the data (g,ξ,λ,μ) which satisfy the equation (3.1) is said to be an η-Ricci soliton on M, in Particular, if μ=0, (g,ξ,λ) is η-Ricci solitons on f-Kenmotsu manifold [11,12] and it is called shrinking, steady or expanding according as λ is negative, zero or positive respectively.

Proposition (3.1): If η -Ricci solitons on f-Kenmotsu manifold (M,ϕ,ξ,η,g), for any X, Y, Z$\epsilon\chi(M)$, the following Relations holds.

$$\nabla_X\xi = f[X-\eta(X)\xi], \quad (3.4)$$

$$\eta(\nabla_X\xi) =0, \quad (3.5)$$

$$R(X,Y)\xi = -(f^2+f')[\eta(Y)X - \eta(X)Y], \quad (3.6)$$

$$\eta(R(X,Y)Z) = -\left(\frac{\tau}{2}+2f^2+2f'\right)gY,Z\eta X-gX,Z\eta Y$$

$$-\left(\frac{\tau}{2}+3f^2-3f'\right)-gY,Z\eta X-gX,Z\eta Y, \quad (3.7)$$

$$(\nabla_X\eta)(Y)=f[g(X,Y)-\eta(X)\eta(Y)], \quad (3.8)$$

$$(\nabla_X\eta)=0, \quad (3.9)$$

$$L_\xi g(X,Y) = 2f[g(X,Y)-\eta X\eta Y, \quad (3.10)$$

where R is the Riemannian curvature tensor field and ∇ denotes is the Levi-Civita connection Associated to g.

$$(L_\xi\phi)(X)= \nabla_\xi\phi X - \phi(\nabla_\xi X), \quad (3.11)$$

which on using of the equation (3.11), in above equation X=ξ, we get

$$(L_\xi\phi)=0,$$

In view of the using equation (3.1), in above equation, we get

$$L_\xi g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) =0. \quad (3.12)$$

Now, taking from the above equation (3.12), in Y=ξ, *we get*

$$L_\xi\eta(X) + 2S(X,\xi) + (2\lambda+2\mu)\eta(X) =0, \quad (3.13)$$

which on using equations (3.13) and (2.10) in above equation, we get

$$L_\xi\eta(X) + [(2\lambda+2\mu) - 4(f^2+f')\eta(X) =0. \quad (3.14)$$

By virtue of the equations (3.1) and (2.4), in above equation, we get

$$(\lambda+\mu) = 2(f^2+f') \quad (3.15)$$

Again, we have

$$(L_\xi\eta)(X)=-\eta(\nabla_\xi X), \quad (3.16)$$

In view of the using equation (3.16), in X=ξ, we get

$$(L_\xi\eta)=0,$$

Again, we have

$$(L_\xi g)(X,Y) =2f[g(X,Y)-\eta(X)\eta(Y)],$$

The (0,2)-tensor field for this proposition we get

$$\omega(X,Y)= g(X,\phi X),$$

Is symmetric and satisfies

$\omega(\phi X,Y)=\omega(X,\phi Y)$, $\omega(\phi X,\phi Y)=\omega(X,Y)$

$(\nabla_X W)(Y,Z)= \eta(Y)g(X,Z)+\eta(Z)g(X,Y)+2\eta(X)\eta(Y)\eta(Z)$, for any X,Y,Z$\epsilon\chi(M)$

Remark: an η -Ricci solitons on f-Kenmotsu manifold (M, ϕ,ξ,η,g) we deduce that

Proposition (3.2): If η -Ricci solitons on f-Kenmotsu manifold (M,ϕ,ξ,η,g) the Para-contact from η-closed and the Nijenhuis Tensor field of the structural endomorphism ϕ identically vanishes.

Proof: (1), The 1-form η is closed indeed, from the above equation (2.6), we gets

$$(d\eta)(X, Y)=g(Y, \nabla_X\xi)-g(X, \nabla_Y\xi), \quad (3.17)$$

Now, taking from the above equations (3.17) and (2.7), we get

$$(d\eta)(X,Y)=0,$$

(ii) The Nijenhuis tensor field associated to ϕ.

$$N_\phi(X,Y)=0,$$

In [11] and [12] the authors proved that on a η-Ricci soliton on f-Kenmotsu manifold (M,ϕ,ξ,η,g) tensor field satisfies. Now, from equation (3.3) and (2.10) as Y=ξ, we get

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$$S(X,\xi)=(-\lambda+2f+\mu)(X). \quad (3.18)$$

which on using equations (3.18) and (2.14), in above equation, we get

$$-2(f^2+2f') = (-\lambda+\mu) \quad (3.19)$$

By virtue of the equations (3.18), (3.19) and (3.15), in above equation, we get

$$\lambda=\mu \quad (3.19)$$

If μ=0, then λ=0,which shows that λ is steady. Thus we can states as follows-

Theorem (3.2): Let (ϕ,ξ,η,g) be η-Ricci solitons on f-Kenmotsu structure on the manifold M and Let (g,ξ,λ,μ) be an η-Ricci Soliton on M.
(i) If the manifold M and Let (M, g) has cyclic tensor

$$\nabla_X S(Y,Z)+\nabla_Y S(Z,X)+\nabla_Z S(X,Y)=0,$$

(ii) If the manifold (M ,g) has cyclic η −recurrent Ricci tensor

$$\nabla_X S(Y,Z)+\nabla_Y S(Z,X)+\nabla_Z S(X,Y)=(f+\mu)f[2g(X,Y)\eta(Z)+2g(X,Z)\eta(Y)+2g(Y,Z)\eta(X)-6\eta(X)\eta(Y)\eta(Z)],$$

Proof: Replacing the expansion of S form

$$\nabla_X S(Y,Z)=(f+\mu)f[g(X,Y)\eta(Z)-2\eta(X)\eta(Y)\eta(Z)+g(X,Z)\eta(Y)], \quad (3.20)$$

By cyclic permutation X, Y and Z, we get

$$\nabla_Y S(Z,X)=(f+\mu)f[g(Y,Z)\eta(X)-2\eta(X)\eta(Y)\eta(Z)+g(Y,X)\eta(Z)], \quad (3.21)$$

And

$$\nabla_Z S(X,Y)=(f+\mu)f[g(Z,X)\eta(Y)-2\eta(X)\eta(Y)\eta(Z)+g(Z,Y)\eta(X)]. \quad (3.22)$$

Adding from equations (3.20), (3.21) and (3.22), we get

$$\nabla_X S(Y,Z)+\nabla_Y S(Z,X)+\nabla_Z S(X,Y)=(f+\mu)f[2g(X,Y)\eta(Z)+2g(X,Z)\eta(Y)+2g(Y,X)\eta(Z)-6\eta(X)\eta(Y)\eta(Z)], \quad (3.23)$$

Corollary (3.3): If η-Ricci soliton on f-Kenmotsu manifold. (M,ϕ,ξ,η,g) having cyclic Ricci tensor or cyclic η-recurrent Ricci tensor, then is no Ricci solitons with potential vector field ξ.

Proposition (3.4): If Let (ϕ,ξ,η,g) be an η-Ricci soliton on f-Kenmotsu structure (M, g) is Ricci-symmetric tensor ∇S=0, then λ=0, is steady.

Proof: If ∇s=0, taking from the above equation (3.3), we get

$$(\nabla_X S)(Y,Z)=(f+\mu)[\eta(Y)(\nabla_X\eta)Z+\eta(Z)(\nabla_X\eta)(Y)], \quad (3.24)$$

By virtue of the equations (3.24) and (2.8), in above equation, we get

$$(\nabla_X S)(Y,Z)=(f+\mu)f[g(X,Y)\eta(Z)-2\eta(X)\eta(Y)\eta(Z)+g(X,Z)\eta(Y)]. \quad (3.25)$$

Taking from the above equation (3.25), Z=ξ,we get

$$(\nabla_X S)(Y,\xi)=(f+\mu)f[g(X,Y)-\eta(X)\eta(Y)], \quad (3.26)$$

where $(\nabla_X S)(Y,\xi)$ =0, $(f+\mu)f$ =0, or [g(X,Y)−$\eta(X)\eta(Y)$] ≠0, f=0, or $f=-\mu$

If f=0, then μ=0. Now, from equation (3.15), we get

$$\lambda=0$$

which shows that λ is steady. Thus we can states as follows-

Corollary (3.5): If η-Ricci soliton on f-Kenmotsu manifold (M,ϕ,ξ,η,g) is Ricci symmetric or has η-recurrent Ricci tensor, then M is with the potential vector field ξ.

In what follows we shall consider η-Ricci soliton on f-kenmotsu manifolds requiring for the curvature to satisfy R(ξ,X).S=0 and S.(ξ,X).R=0, respectively.

Theorem (3.6): If η-Ricci soliton on f-Kenmotsu structure (ϕ,ξ,η,g) on the manifold M, then satisfying relations R(ξ,X).S=0, as f=0, then λ is steady.

Proof: Letus suppose that R(ξ,X).S=0, then we have

$$S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)=0, \quad (3.27)$$

Replacing the expression of S from equation (2.8) and from the symmetric of R, we get

$$(\tau+5f^2-f')[g(X,Y)S(\xi,Z)+g(X,Z)S(Y,\xi)]-(f^2-5f')[\eta(Y)S(X,Z)+\eta(Z)S(Y,X)]=0. \quad (3.28)$$

Now, from above equations (3.28) and (3.3), we get

$$[(\lambda+\mu)(\tau+5f^2-f')-(f^2+5f')(\lambda-f)][g(X,Y)\eta(Z)-g(X,Z)\eta(Y)]-2(f^2+5f')(f+\mu)\eta(X)\eta(Y)\eta(Z)=0. \quad (3.29)$$

Taking from equation (3.29), in X=Y=ξ,we get

$$-2(f^2+5f^1)(f+\mu)\eta(Z)=0. \quad (3.30)$$

Now, from equation (3.30), we get

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$f^2 + 5f'$=0, or f= - μ

Case 1, if f=0, then μ=0, case 2, if $f^2 + 5f'$=0, if f=0, then f'=0,

Theorem (3.7): If η-Ricci soliton on f-Kenmotsu manifold is also η-Ricci soliton on f-Kenmotsu structure on the manifold (M g, ξ, λ, μ) and (ϕ, ξ, η, g) is with potential vector field ξ and satisfying R(ξ, X).S=0 and S(ξ, X).R=0. If λ=0, τ=0, or if f=0, f'=0, μ=0,then λ is stesdy.

Proof : Letus suppose that R(ξ, X).S=0 and S(ξ, X).R=0, Then we have

S(X,R(Y,Z)W) ξ - S($\xi, R(Y,Z)W$)X+S(X,Y)R(ξ, Z)W-S(ξ, Y)R(X,Z)W +S(X,Z)R(Y, ξ)W-S(ξ, Z)R(Y,X)W+S(X,W)R(Y,Z) ξ - S(ξ, W)R(Y,Z)X=0, (3.31)

Now, taking the inner product of above equation with ξ, we get

S(X,R(Y,Z)W)-$\eta(X)S(\xi, R(Y,Z)W) + S(X,Y)\eta(R(\xi,Z)W) - S(\xi,Y)\eta(R(X,Z)W)$
$+S(X,Z)\eta(R(Y,\xi)W) - S(\xi,Z)\eta(R(Y,X)W)$

$+S(X,W)\eta(R(Y,Z)\xi) - S(\xi,W)\eta(R(Y,Z)X)$ =0. (3.32)

By virtue of the equations (3.32) and (3.7), above equation, we get

$$S(X, R(Y, Z)W)- \eta(X)S(\xi,R(Y,Z)W) + S(X,Y)g(Z,W) - S(X,Y)\eta(X)\eta(W) + \left(\frac{\tau}{2} + 3f^2 - 3f'\right)[S(X,Y)g(Z,W) - S(X,Y)\eta(W)\eta(Z) + g(Z,W)\eta(X) - g(X,W)\eta(Z) + S(X,Z)\eta(Y)\eta(W) - g(Y,W) + S(\xi,Z)g(X,W)\eta(Y) + S(\xi,Z)g(Y,Z)\eta(X) + 2\eta(Y)\eta(Z) + g(Z,X)\eta(Y) + g(Y,X)\eta(Z)] + \left(\frac{\tau}{2} + 2f^2 + 2f'\right)[-S(\xi,Y)g(Z,W)\eta(X) + g(X,W)S(\xi,Y)\eta(Z)] + S(X,Z)\eta(Y)\eta(W) - S(X,Z)g(Y,W) - S(\xi,Z)g(X,W)\eta(Y) - S(\xi,Z)g(Y,W)\eta(X)] - \left(\frac{\tau}{2} + 2f^2 + 2f'\right)\left(\frac{\tau}{2} + 2f^2 + 2f'SξWgZ,XηY+gY,XηZ\right)=0, \quad (3.33)$$

which on using equations (3.33) and (3.3) in Putting X=Y=ξ,we get

$$(\lambda + \mu)\ g\ (Z,W)-2(- f + \mu)\ \eta(Z)\eta(W) + \left(\frac{\tau}{2} + 3f^2 - 3f'\right)[(-f + \lambda)\{g(Z,W) + 4\eta Z\eta W + (\lambda + \mu)\{g(Z,W) + 4\eta(Z)\eta(W)\} - \eta(Z)\eta(W) - \eta(W) + 4\eta(Z)] - \left(\frac{\tau}{2} + 2f^2 + 2f'\right)(-f + \mu)[-g(Z,W) + \eta(Z)\eta(W)] + \left(\frac{\tau}{2} + 2f^2 + 2f'\right)[-g(Z,W) + \eta(Z)\eta(W)] - \left(\frac{\tau}{2} + 2f^2 + 2f'2[(\lambda+\mu)\eta Z\eta W+\eta(Z)]\right)=0. \quad (3.36)$$

Using from the above equation (3.34) in Z=W=ξ, we get

$$\lambda - \mu + f + \left(\frac{\tau}{2} + 3f^2 - f'\right)[5(\lambda + \mu) + 2] - \left(\frac{\tau}{2} + 2f^2 + 2f'\right)(1 + \lambda + \mu)=0, \quad (3.35)$$

If μ=0, f=0, then f'=0, $\lambda = \frac{\tau + \frac{\tau^2}{4}}{1 - \frac{5}{2}\tau - \frac{\tau^4}{4}}$.

If τ=0, then λ=0,

which shows that λ is steady. Thus we can states as follows-

Example (3.8): 3-dimensional η-Ricci soliton on f-Kenmotsu manifold with the Schouten-van Kanpen connection we consider the 3-dimensional manifold M={(X,Y,Z) ϵR^3, Z $\neq$ 0}, where (X,Y,Z) are the Standard coordinates in R^3. the vector fields

$$e_1 = Z^2 \frac{d}{dX}\ ,\ e_2 = Z^2 \frac{d}{dY}, e_3 = \frac{d}{dZ}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$g(e_1, e_3)= g(e_2, e_3)= g(e_1, e_2)$=0, $g(e_1, e_1)= g(e_2, e_2)=g(e_3, e_3)$=1.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any Z $\epsilon\chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3)$ =0, then using linearity of ϕ and g, we have

$\eta(e_3) = 1, \phi^2 Z = -Z + \eta(Z)e_3$,
$g(\phi Z, \phi W)$= g(Z,W)-$\eta(Z)\eta(W)$,

for any Z,W $\epsilon\chi(M)$. Now by direct computations we obtain

$[e_1, e_2]$ =0, $[e_2, e_3] = -\frac{2}{Z}e_2, [e_1, e_3] = -\frac{2}{Z}e_1$,

The Riemannian connection of the metric tensor g is given by the Koszul's formula which is

2 $g(\nabla_X Y, Z)$=Xg(Y,Z)+Yg(Z,X)-2g(X,Y)-g(X,[Y,Z])- g(Y,[X,Z])+g(Z,[X,Y]). (3.36)

Using from the above equation (3.36), we get

2 $g(\nabla_{e_1} e_3, e_1)$=2$g(-\frac{2}{Z}e_1, e_1)$, 2 $g(\nabla_{e_1} e_3, e_2)$=0, and 2 $g(\nabla e_1, e_3, e_3)$=0,

Hence $\nabla_{e_1} e_3 = -\frac{2}{Z}e_1$, Similary $\nabla_{e_2} e_3 = -\frac{2}{Z}e_2$, $\nabla_{e_3} e_3$=0, further yields

$\nabla_{e_1} e_2 = 0$, $\nabla_{e_1} e_1 = \frac{2}{z} e_3$, $\nabla_{e_2} e_2 = \frac{2}{z} e_3, \nabla_{e_2} e_1 = 0$, $\nabla_{e_3} e_2 = 0$, $\nabla_{e_3} e_1 = 0$. (3.37)

From (3.37), we see that the manifold satisfies

$$\nabla_X \xi = f[X - \eta(X)\xi], \text{ for } \xi = e_3$$

where f=$-\frac{2}{z}$,

Hence we conclude that M is an η-Ricci soliton on f-Kenmotsu manifold. Also $f^2 + f' \neq 0$, known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$R(e_1, e_2)e_3 = 0$, $R(e_2, e_3)e_3 = -\frac{\epsilon}{z^2} e_2$, $R(e_1, e_2)e_3 = -\frac{\epsilon}{z^2} e_1, R(e_1, e_2)e_2 = -\frac{4}{z^2} e_1$,

$R(e_3, e_2)e_2 = -\frac{\epsilon}{z^2} e_3$, $R(e_1, e_3)e_2 = 0$, $R(e_1, e_2)e_1 = \frac{4}{z^2} e_2, R(e_2, e_3)e_1 = 0$, $R(e_1, e_3)e_1 = \frac{6}{z^2} e_3$.

The Schouten-Van Kampen connection on M is given by

$\nabla_{e_1} e_3 = \left(-\frac{2}{z} - f\right) e_1, \nabla_{e_2} e_3 = \left(-\frac{2}{z} - f\right) e_2$,
$\nabla_{e_3} e_3 = -f(e_3 - \xi), \nabla_{e_1} e_2 = 0$,
$\overline{\nabla}_{e_2} e_2 = \frac{2}{z}(e_3 - \xi), \nabla_{e_3} e_2 = 0$, $\nabla_{e_1} e_1 = \frac{2}{z}(e_3 - \xi), \nabla_{e_2} e_1 = 0$, $\nabla_{e_3} e_1 = 0$, $\nabla_{e_i} e_j = 0$. (3.38)

From (3.38), we can see that $\nabla_{e_i} e_j = 0$, $(1 \leq i, j \leq 3)$ for $\xi = e_3$ and f= $-\frac{2}{z}$.

` Hence M is a 3-dimensional η -Ricci soliton on f-Kenmotsu manifold with the Schouten-Van Kampen connection.also using (3.38), it can be seen that R=0.Thus the manifold M is a flat manifold with respect to the Schouten-Van Kampen connection.Since a flat is a Ricci-flat manifold with respect to the Schouten-Van Kampen connection, the manifold M is both a Projectively flat and a Conharmonically flat 3-dimensional η -Ricci soliton on f-Kenmotsu manifold with respect to the Schouten-Van Kampen connection. So, from theorem 1 and theorem 2 is η-Einstein manifold with respect to the Levi-Civita connection.

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