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Lacunary d−statistical α−boundedness

Priya Gupta
Department of Mathematics, Arya P. G. College, Panipat-132103 (INDIA)

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ABSTRACT: In this paper, we introduce and examine the concept of lacunary d−statistical α−convergence and lacunary d−statistical α−boundedness and establish the realtion between them. Finally, we give a general description of inclusion between two arbitrary lacunary methods of d−statistical α−convergence.

I. INTRODUCTION AND PRELIMINARIES

The idea of statistical convergence which is, in fact, a generalization of the usual notion of con- vergence was introduced by Fast [15] and Steinhaus [28] independently in the same year 1951 and since then several generalizations and applications of this concept have been investigated by various authors namely Bhardwaj et al. ([1], [2], [3], [4],[5]), Connor [10], Et [11], Et et al.([12], [13], [14]), Fridy [18], Fridy and Orhan [19], Mursaleen and Mohiuddine [23], Mursaleen [24], Rath and Tripathy [25], Salat [27], and many others.

The idea of statistical convergence depends upon the density of subsets of the set N of natural numbers. The natural density δ(K) of a subset K of the set N of natural numbers is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|$$

where $|\{k \le n : k \in K\}|$ denotes the number of elements of K not exceeding n. Obviously, we have $\delta(K) = 0$ provided that K is a finite set.

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathrm{N} : |x_k - L| \ge \varepsilon\}) = 0,$$

$$i.e. \lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

In this case we write S lim $x_k = L$. Since lim $x_k = L$ implies S lim $x_k = L$, statistical convergence may be considered as a regular summability method. The set of all statistically convergent sequencesis denoted by S.

Following Freedman et al. [17], by a lacunary sequence $\theta = \{k_r\}_{r=0}^{\infty}$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and we let $h_r = k_r - k_{r-1}$. Sums of the form

$\sum_{i=k_{r-1}+1}^{k_r} |x_i| = \sum_{i\in I_r} |x_i|$ will be written for convenience as $\sum_{I_r} |x_i|$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

There is a strong connection [17] between the space $|\sigma_1|$ of strongly Cesàro summable sequences:

$$|\sigma_1| = \{x = (x_k) : \text{there exists L such that } \frac{1}{n}\sum_{k=1}^{n} |x_k - L| \to 0\}$$

and the sequence space N_θ, which is defined by

$$N_\theta = \{x = (x_k) : \text{there exists L such that } \frac{1}{h_r}\sum_{I_r} |x_k - L| \to 0\}.$$

Fridy and Orhan [19] introduced and studied a concept of convergence, called lacunary statistical convergence, that is related to statistical convergence in the same way that N_θ is related to $|\sigma_1|$.

Definition 1.1 Let θ be a lacunary sequence. The number sequence $x = \{x_k\}$ is lacunary statistical convergent or S_θ-convergent to L provided that for every $\epsilon > 0$, $\lim \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$.
In this case, we write $S_\theta - \lim x = L$ or $x_k \to L(S_\theta)$, and we define $S_\theta = \{x :$ for some $L, S_\theta - \lim x = L\}$.

Statistical convergence of order α ($0 < \alpha \leq 1$) was introduced by Ç olak [8], and also indepen- dently by Bhunia et al. [6], using the notion of natural density of order α (where n is replaced by n^α in the denominator in the definition of natural density). It was observed in ([6], [8]) that the behaviour of this new kind of convergence was not exactly parallel to that of statistical convergence. For a detailed account of many more interesting investigations concerning statistical convergence of order α, one may refer to ([2], [9], [12]) and [26], where many more references can be found.
Let α be any real number such that $0 < \alpha \leq 1$.
The α− density of a set $K \subset N$ is defined by

$$\delta^\alpha(K) = \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : k \in K\}|$$

provided this limit exists. Note that α−density of any set reduces to its natural density in case α = 1. In case of natural density, it is well known that $\delta(K) + \delta(N - K) = 1$. But this result remains no longer true in case of α− density, i.e., $\delta^\alpha(K) + \delta^\alpha(N - K) = 1$ does not hold, in general. Moreover, as in the case of natural density, α− density of a finite set is also zero.

If K has zero $\alpha-$ density for some $\alpha \in (0, 1)$, then it has zero natural density. But the converse need not be true, in the sense that a set having zero natural density may have non-zero $\alpha-$ density for some $\alpha \in (0, 1)$. For example, if we take $K = \{1, 4, 9, \ldots\}$ then $\delta(K) = 0$ but $\delta^\alpha(K) = \infty$ for any $\alpha \in (0, \frac{1}{2})$.

Let $0 < \alpha \leq 1$. A number sequence $x = (x_k)$ is said to be statistically convergent of order α to L, if for each $\varepsilon > 0$

$$\delta^\alpha(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n^\alpha}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

and we write $S^\alpha - \lim x_k = L$. The set of all statistically convergent sequences of order α is denoted by S^α. In case $\alpha = 1$, the statistical convergence of order α reduces to the statistical convergence.

The concept of statistical boundedness was given by Fridy and Orhan [20] as follows:

The real number sequence x is statistically bounded if there exists a number $B \geq 0$ such that $\delta(\{k : |x_k| > B\}) = 0$.

It can be shown that every bounded sequence is statistically bounded, but the converse is not true. For this consider a sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} k, & \text{if } k \text{ is a square} \\ 1, & \text{if } k \text{ is not a square} \end{cases}$$

Clearly $x = (x_k)$ is not a bounded sequence, but it is statistically bounded.

Bhardwaj and Gupta [4] generalized the concept of statistical boundedness by introducing theconcept of α−statistical boundedness as follows :

The real number sequence $x = (x_k)$ is statistically bounded of order α $(0 < \alpha \leq 1)$ if there is a number $B \geq 0$ such that

$$\delta^{\alpha}(\{k \in \mathbb{N} : |x_k| \geq B\}) = 0,$$

i.e.,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}}|\{k \leq n : |x_k| \geq B\}| = 0.$$

The sets of all statistically bounded and statistically bounded sequences of order α are denoted by $S(b)$ and $S^{\alpha}(b)$, respectively.

Definition 1.2 Let $\theta = \{k_r\}$ be a lacunary sequence. The number sequence $x = \{x_k\}$ is said to be lacunary statistical bounded or S_θ-bounded if there exists $M > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : |x_k| > M\}| = 0,$$

i.e.,

$$\delta^{\theta}(\{k \in \mathbb{N} : |x_k| > M\}) = 0,$$

For a given lacunary sequence $\theta = \{k_r\}$, $S_\theta(b)$ denotes the set of all S_θ-bounded sequences. Obvi-ously, $S_\theta(b)$ is a linear space with respect to co-ordinatewise addition and scalar multiplication.

In the present paper we introduce the concept of lacunary d−statistical α−convergence and lacunary d−statistical α−boundedness and establish the realtion between them.

II. MAIN RESULTS

Definition 2.1 Let (X, d) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. The sequence $x = (x_k)$ in X is said to be $S^{\alpha,d}$−convergent or lacunary d−statistically α−convergent if there is areal number $a \in X$ such that

$$\lim_{r\to\infty} \frac{1}{h_r^{\alpha}}|\{k \in I_r : d(x_k, a) \geq \epsilon\}| = 0.$$

where $B_\epsilon(a)$ is the open ball of radius ϵ and centre a. In this case, we write $S_\theta^{\alpha,d} - \lim x_k = a$. The set of all lacunary d−statistically α−convergent sequences will be denoted by $S_\theta^{\alpha,d}$

If $\theta = (2^r)$ and $\alpha = 1$, then lacunary d − statistical α convergence reduces to d − statistical convergence in a metric space which was introduced by Kucukaslan et. al. [21].

Definition 2.2 Let (X, d) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. The sequence $x = (x_k)$ in X is said to be lacunary d statistically α bounded if there is a real number a X and a real number B such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : d(x_k, a) \geq B\}| = 0.$$

The set of all lacunary d statistically α bounded sequences will be denoted by $S^{\alpha,d}(b)$. If $\theta = (2^r)$ and $\alpha = 1$, then lacunary d− statistical α boundedness reduces to d statistical boundedness in a metric space which was introduced by Kucukaslan et. al. [22].

Theorem 2.3 Every lacunary d statistically α convergent sequence is lacunary d statistical α bounded; but the converse is not true.

Proof. Let $x = (x_k)$ be a lacunary d−statistically α−convergent sequence and ε > 0 be given. Then there exist a ∈ X such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : d(x_k, a) \geq \varepsilon\}| = 0.$$

Now for any real number B with $B > \varepsilon$, we have

$$|\{k \in I_r : d(x_k, a) \geq B\}| \leq |\{k \in I_r : d(x_k, a) \geq \varepsilon\}|$$

and consequently, result follows. To show the strictness of the inclusion, choose $\theta = (2^r)$; $X = \mathbb{R}$, $d(x, y) = |x - y|$, $\alpha = 1$ and consider a sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1 & k = n^2 \\ 0 & k \neq n^2 \end{cases} \quad n \in \mathbb{N}.$$

It is clear that $x = (x_k)$ is lacunary d−statistically α−convergent to 0 but it is not convergent.

Theorem 2.4 Every bounded sequence is lacunary d statistically α bounded; but the converse is not true.

Proof. Let $x = (x_k)$ in (X, d) be a bounded sequence. Then there exists a real number a ∈ X and a real number B such that $|\{k \in I_r : d(x_k, a) \geq B\}| = 0$ for all r ∈ N and so,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : d(x_k, a) \geq B\}| = 0.$$

To show the strictness of the inclusion, choose $\theta = (2^r)$; $X = \mathbb{R}$, $d(x, y) = |x - y|$, $\alpha = 1$ and consider a sequence $x = (x_k)$ by

$$x_k = \begin{cases} k & k = n^2 \\ -1 & k \neq n^2 \end{cases} \quad n \in \mathbb{N}.$$

It is clear that $x = (x_k)$ is not bounded but, it is d−statistically bounded.

Theorem 2.5 *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$,*

(i) if $\liminf \left(\frac{h_r}{l_r}\right)^{\alpha} > 0$ then $S^d_{\theta',\alpha} \subset S^d_{\theta,\alpha}$

Proof. (i) Suppose that $I_r \subset J_r$ for all r ∈ N and given condition holds. For given ε > 0, we have

$$\{k \in J_r : d(x_k, a) \geq \varepsilon\} \supset \{k \in I_r : d(x_k, a) \geq \varepsilon\}$$

and so

$$\frac{1}{(l_r)^{\alpha}}|\{k \in J_r : d(x_k, a) \geq \varepsilon\}| \geq \frac{1}{(l_r)^{\alpha}}|\{k \in I_r : d(x_k, a) \geq \varepsilon\}|$$
$$= \frac{(h_r)^{\alpha}}{(l_r)^{\alpha}}\frac{1}{(h_r)^{\alpha}}|\{k \in I_r : d(x_k, a) \geq \varepsilon\}|$$

for all $r \in \mathbb{N}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$ and $l_r = s_r - s_{r-1}$. Now taking the limit as $r \to \infty$ in the last inequality and using condition, we get the result.

(ii) Let $x = (x_k) \in S^d_{\theta,\alpha}$ and given condition holds. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\frac{1}{l_r}|\{k \in J_r : d(x_k, a) \geq \varepsilon\}| = \frac{1}{l_r}|\{s_{r-1} < k \leq k_{r-1} : d(x_k, a) \geq \varepsilon\}|$$
$$+ \frac{1}{l_r}|\{k_{r-1} < k \leq k_r : d(x_k, a) \geq \varepsilon\}|$$
$$+ \frac{1}{l_r}|\{k_r < k \leq s_r : d(x_k, a) \geq \varepsilon\}|$$
$$\leq \frac{k_{r-1} - s_{r-1}}{l_r} + \frac{s_r - k_r}{l_r} + \frac{1}{l_r}|\{k \in I_r : d(x_k, a) \geq \varepsilon\}|$$
$$\leq \frac{l_r - h_r}{l_r} + \frac{1}{l_r}|\{k \in I_r : d(x_k, a) \geq \varepsilon\}|$$
$$\leq \left(\frac{l_r}{h_r} - 1\right) + \frac{1}{h_r}|\{k \in I_r : d(x_k, a) \geq \varepsilon\}|.$$

Since $\lim_{r\to\infty} \frac{l_r}{h_r} = 1$ and $x = (x_k) \in S^d_{\theta,\alpha}$, so that the first and second term on right hand side of above inequality tend to 0 as $r \to \infty$. This implies that $S^d_{\theta,\alpha} \subset S^d_{\theta',\alpha}$.

Corollary 2.6 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$,

(i) if $\liminf \frac{h_r}{l_r} > 0$ then $S^d_{\theta'} \subset S^d_{\theta}$

(ii) if $\lim_{r\to\infty} \frac{l_r}{h_r} = 1$ then $S^d_{\theta} \subset S^d_{\theta'}$.

Theorem 2.7 *Let (X, d) be a metric space and let $0 < \alpha \leq \beta \leq 1$ be given. If a sequence $x = (x_k)$ in (X, d) is lacunary $d-$ statistically convergent of order α, then it is lacunary $d-$ statistically convergent of order β, i.e., $S^d_{\theta,\alpha} \subset S^d_{\theta,\beta}$.*

Proof. Let $x = (x_k) \in S^d_{\theta,\alpha}$. Then for $\varepsilon > 0$, there exists $a \in X$ such that

$$\lim_{r\to\infty}\frac{1}{h_r^{\alpha}}|\{k \in I_r : d(x_k, a) \geq \varepsilon\}| = 0.$$

The result follows in view of the following inequality

$$\frac{1}{h_r^{\beta}}|\{k \in I_r : d(x_k, a) \geq \varepsilon\}| \leq \frac{1}{h_r^{\alpha}}|\{k \in I_r : d(x_k, a) \geq \varepsilon\}|.$$

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