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Charactarization of Quasi-Idempotents in Signed Singular Mapping Transformation Semigroup

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ABSTRACT: This paper characterized the quasi-idempotent elements and derive a formula for the overall count of quasi-idempotent elements in signed singular mapping $Q(SSingO_n)$ and signed symmetric inverse semigroup order-preserving $Q(SSO_n)$ of full transformation semigroup, $[\alpha \in Q(SSingO_n),\ Q(SSO_n)\ \ni\ \forall\, i, j \in X_n, |i| \leq |j\,| \Longrightarrow |\alpha(i)| \leq |\alpha(j)|]$.

KEYWORDS: dempotent, semigroup, signed full transformation semigroup, quasi-idempotents signed singular mapping, signed symmetric, order-preserving mapping.

I. INTRODUCTION

Let $Z_n = \{1,2,3, \dots, n\}$ and $Z_n{}^* = \{\pm 1, \pm 2, \pm 3, \dots, \pm n\}$ be finite sets. A map $\alpha: Z_n \longrightarrow Z_n{}^*$ is called a signed

transformation on Z_n. The collection of all signed transformations on Z_n together with composition forms a semigroup called the signed transformation semigroup on the finite set Z_n. We represent this semigroup by ST_n. ST_n, will be called a signed full transformation on Z_n since $dom(\alpha) = Z_n$ and $im(\alpha) \subseteq Z_n{}^*$.

The concept of signed transformations was initiated in Richard (2008). His study on the signed full transformation semigroup ST_n, was an extension to the work of James and Kerber (1981) who studied the signed symmetric group. Since then, researchers have made tremendous headway in the study of signed transformations. They have studied this semigroups with respect to its algebraic and combinatorial properties. On the other hand, the collections of the signed transformations with the composition have also been objects of consideration by algebraist.

Mogbonju and Azzez (2018), have considered various signed transformation seimgroups in their work. They considered signed order preserving transformation, partial order-preserving and order-preserving injective transformations. In all of these studies, their concern is on the combinatorial aspects of the semigroups. They deduced formulas that count the numbers of elements in the semigroup for any natural number n. Mogbonju et al. (2019) undertook an extension to Mogbonju and Azzez (2018). Their study was on the signed singular self maps on the finite set $Z_n{}^*$ $(SSing_n)$. He found the cardinality of $SSing_n$ and the number of idempotents contained in it.

Tal et al (2022) studied the work performed by the elements of T_n. They characterised maps in T_n that attain maximum and minimum work as an extension to the work of Imam and Tal (2019) on the classical full transformations T_n and a general study of semigroups order-decreasing D_n was initiated by Umar (1996), he showed that the order of D_n is n!.

Let $Z_n = \{1,2,3, \dots, n\}$ and $Z_n{}^* = \{\pm 1, \pm 2, \pm 3, \dots, \pm n\}$. Then it is not difficult to see that for the signed semigroup which can be found in Mogbonju and Azzez (2018), Mogbonju et.al (2019a) and Moggbonju et.al (2019b):

$$|\,ST_n\,| = 2n^n + n^n(2^n - 2)$$
$$|\,SP_n\,| = (2n + 1)^n$$
$$|SIO_n| = \sum_{k=0}^{n} \binom{n}{k} \binom{n\ +\ k}{n}$$
$$|\,SPT_n \backslash SS_n| = (2n + 1)^n - 2^n n$$
$$|\,SSing_n| = 2^n(n^n - n!)$$
$$|\,SO_n| = 2^n \binom{2n\ -\ 1}{n\ -\ 1}$$

The number of idempotent elements in signed semigroup, ST_n was computed by Mogbonju and Azzez (2018) as $|E(SO_n)| = \frac{1}{3n+1}\binom{4n}{n}$ and for $|E(SSing_n)|$ and $|E(SP_n)|$ were obtained by Mogbonju et.al (2019a) and Mogbonju et.al (2019b) as $|E(SSing_n)| = \frac{n^2(n^2-1)}{3}$.

In this paper, we use the ideals proposed by Mogbonju and Azzez (2018), Mogbonju et al. (2019a), and Mogbonju et al. (2019b) to extent the work of Mogbonju et al. (2019b) by characterized the quasi-idempotent elements in signed singular mapping, $Q(SSing_n)$, signed symmetric inverse semigroup $Q(SS_n)$ and derive a formula for the overall count of quasi-idempotent elements in signed singular mapping of full transformation order-preserving $Q(SSingO_n)$.

II PRELIMINARIES

2.1 Semigroup

A groupoid is a pair consisting of a non-empty set S and a binary operation * defined on S. we say that groupoid $(S,*)$ is a semigroup if the operation * is associative in S, that is to say, if, for all x, yand z in S, the equality $(x*y)*x = x*(y*z)$ holds if in a semigroup S the binary operation has the property that, $\forall x, y, x \in S, xy = yx$ we say that S is a commutative semigroup. If a semigroup S contains an element 1 with the property that, for all $\forall x \in S,\ x1 = 1x = x$ the S is called a semigroup with identity, and the element 1 is called the identity element of S.

Theorem 2.1 (Howie (1995)). A semigroup S has at most one identity.
Proof.
See Howie (1995)
If S is a semigroup, which has no identity element, then it is very easy to adjoin an extra element 1 to S (to form a monoid out of S) given that $1s = s1 = s\ \forall s \in S$, and $1s = s1 = s$ for all $s \in S$, and $11 = 1$, it is then easy to see that $S\cup\{1\}$ becomes a monoid. Given monoid, denoted by S^1, is defined by

$$S^1 = \begin{cases} S & \text{if S has identity} \\ S\cup\{1\} & \text{otherwise} \end{cases}$$

and called a semigroup with identity adjoined if necessary. If a semigroup S with at least two elements contains an element 0 given that, $\forall x \in S, 0x = x0 = x = 0$, then S is called semigroup with zero and the element 0 as the zero element of S.
By analogy with case of S^0, for any semigroup S, we defined

$$S^0 = \begin{cases} S & \text{if S has identity} \\ S\cup\{0\} & \text{otherwise} \end{cases}$$

and refers to S^0 as the semigroup obtained from S by adjoining a zero if necessary.

2.2 Subsemigroup and Ideals

A non – empty subset T of a semigroup S is called a subsemigroup of S if it is closed with respect to multiplication that is, if $\forall x, y \in T,\ xy \in T$. If A and B are subset of a semigroup S, then we write AB to mean the set $\{ab: a \in A \text{ and } b \in B\}$ and that $A^2 = a_1a_2: a_1a_2 \in A$.

The condition of closure in the definition of subsemigroup can be stated as $T^2 \subseteq T$.
A subsemigroup of S which is a group with respect to the multiplication inherited from S is called a subgroup of S.

2.3 Regular semigroups

An element a of a semigroup S is called regular if there exist $x \in S$ given that $xax = a$. The semigroup S is called regular if all its elements are regular. That is if $[(\forall a \in S)\exists x \in S \ni axa = a]$.

2.4 Ideal and Green's relations

The ideal theory naturally leads to the study of certain equivalence relations on a semigroup. Green first introduced these equivalence connections in 1951, and they have been crucial in the development of semigroup theory. Since their founding, the have become widely used tools for studying the semigroup structure. If a is an element in a semigroup S, the sets $S^1a = Sa\cup\{a\}$, $aS^1 = aS\cup\{a\}$ and $aS^1 = SaS\cup Sa\cup aS\cup\{a\}$, are left, right and two sided ideals of S respectively. These are respectively the smallest left, right and two sided Ideals of S containing a. We shall call them principal left, right and two-sided ideals of S generated by a respectively. For any two $a, b \in S$, we define the equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ on S by;
$a\,\mathcal{L}\,b$ if and only if $S^1a = S^1b$
$a\,\mathcal{R}\,b$ if and only if $aS^1 = bS^1$
$a\,\mathcal{J}\,b$ if and only if $S^1aS^1 = S^1bS^1$
$\mathcal{H} = \mathcal{L}\cap\mathcal{R}$ and $\mathcal{D} = \mathcal{L}o\mathcal{R}$.

We refer to these five equivalences as Green's relation (Howie, 1995). This is enough to established that $T_n = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3 \cup \cdots \cup \mathcal{J}_{n-1}$, where $\mathcal{J}_1\mathcal{J}_2\mathcal{J}_3\cdots\mathcal{J}_{n-1}$ are $\mathcal{J}$ −classes with $\mathcal{J}_r = \{\alpha \in T_n : |im(\alpha)| = r\}$.

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Also as a consequence of this, we see that, the $\mathcal{J}$ −classes in T_n are $\mathcal{J}_r$ and the number of $\mathcal{L}$ −classes is the number of distinct subset of Z_n cardinality r, that is, the binomial coefficient $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. The number of $\mathcal{R}$ −classes is the number of equivalence on Z_n having r classes, that is, the stirling number of the second kind $S(n,r)$ defined recursively as $S(n,r) = S(n-1,r-1) + rS(n-1,r)$ with boundary condition $S(n,1) = S(n,n) = 1$, Also, $S(n,n-1) = \frac{n(n-1)}{2}$ and $S(n,2) = 2^{n-1}$. Therefore, a $\mathcal{J}$ −class $\mathcal{J}_r$ of T_n is visualized as an egg box in which the α −classes are the column, the $\mathcal{R}$ −classes are the rows and the $\mathcal{H}$ −classes are the cells. The number of cells is $\binom{n}{r} \times S(n,r)$ and each cell contains $r!$ elements.

Propositions 2.1 (Howie (1995)).
Let $\alpha, \beta \in S$.
1. $\alpha \mathcal{L} \beta$ if and only if $Im(\alpha) = Im(\beta)$
2. $\alpha \mathcal{R} \beta$ if and only if $Ker(\alpha) = Ker(\beta)$
3. $\alpha \mathcal{L} \beta$ if and only if $|Im(\alpha)| = |Im(\beta)|$
4. $\mathcal{D} = \mathcal{J}$.

Proposition 2.2
Let $S = P_n \backslash S_n$ if $\alpha \in S, n \geq 0$, then $|S| = (n+1)^n - n!$
Proof. See Garba (1990)

Proposition 2.3: Let $S = P_n$, if $\alpha \in S, n \geq 0$, then $|S| = (n+1)^n$
Proof.
Each element $\alpha \in P_n$ is uniquely defined by $\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ r_1 & r_2 & \cdots & r_n \end{pmatrix}$, where $r_i = \alpha(i)$ if $i \in dom(\alpha)$ and $r_i = \phi$ if $i \in dom(\alpha)$ again the element r_i can be independently chosen from the set $\mathcal{N} \cup \{\Phi\}$. Hence the product rule implies $|P_n| = (n+1)^n$ elements.

Theorem 2.1:
An element $\alpha \in O_n$ is a quasi-idempotent if and only if the image of each non-stationary block of α is contained in a stationary block of α, Imam et. al (2023).
Proof.
See , Imam et. al (2023).

Definition: 2.1 Consider $Z_n = \{1,2,\cdots n\}$ and $\alpha: dom(\alpha) \subseteq Z_n \longrightarrow im(\alpha) \subseteq Z_n$ be full transformation. We recall that, $dom(\alpha)$ and $im(\alpha)$ stands for domain of mapping α and image set or range of the mapping α respectively, as stated by Imam et. al (2023). Let $\alpha \in T_n$ be denoted by $\alpha = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix}$, where $T_1, T_2, \cdots, T_n$ are pair wise disjoint subsets of X_n called the blocks of α with $\alpha(T_i) = t_i$. We can easily see that, the $im(\alpha) = \{t_1, t_2, \cdots t_n\}$ and $T_1 \cup T_2 \cup \cdots T_n = X_n$, if $t_i = T_i$ we say that T_i is stationary otherwise it is non-stationary.
Definition 2.2: A transformation $\alpha \in ST_n$ is order-preserving if $\forall\, i,j \in X_n$, $|i| \leq |j| \Longrightarrow \alpha(i) \leq \alpha(j)$ Mogbonju and Azze (2018).

Definition 2.3: A transformation $\alpha \in ST_n$ is order-decreasing if $\forall |i| \in X_n$, $|\alpha i| \leq |i|$

Definition 2.4: For a map $\alpha \in ST_n$, we defined $h(\alpha) = |dom(\alpha)|$ and $f(\alpha) = |\{i \in dom(\alpha): |i\alpha| = i\}|$ to be the height and fix of α respectively.

Definition 2.5: An element $\alpha \in ST_n$ is called self-inverse element if $\alpha^2 = i$ (the identity elements). We use $I(ST_n)$ to denote the set of all self-inverse in ST_n.

III. MATERIAL AND METHODS

3.1 Matrix Notation

Let $\alpha \in SO_5$: By inserting ± 1 in the entry of n by n matrix to denote $i \longrightarrow \pm j$, this element can be represent in matrix form .
Example:

If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -3 & -4 & 2 & 5 & 1 \end{pmatrix} \in SSing_n$ it can be written in matrix as

$$\alpha = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in SSing_5$$

3.2 Enumeration of element in $Q(SSingO_n)$ and $Q(SSO_n)$

We obtain the elements of $Q(SSingO_n)$ for small values of $n = 1,2,3,4$ by taking in to account only order-preserving.

For $n = 1$ Table 3.1 Elements of height 1 in $Q(SSO_1)$

$\mathcal{J}_1 Q(SSO_1)$	$\{-1\}$
± 1	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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$|Q(SSO_1)| = |\mathcal{J}_1(Q(SSO_1))| = 1$

For $n = 2$ Table 3.2. Elements of height 1 in $Q(SSingO_2)$ and $Q(SSO_2)$

$\mathcal{J}_2 Q(SSingO_2)$	$\{-1,1\}$	$\{-1,-1\}$
± 1	$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

For $n = 2$ Table 3.3. Elements of height 2 in $Q(SSingO_2)$ and $Q(SSO_2)$

$\mathcal{J}_2 Q(SSingO_2)$	$\{\mathbf{-1,2}\}$	$\{\mathbf{-1,-2}\}$	$\{\mathbf{1,-2}\}$	$\{2,-2\}$	$\{-2,-2\}$
$\pm 1 \pm 2$	$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{-2} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{-2} \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$

$|Q(SSingO_2)| = |\mathcal{J}_1(Q(SSinO_2))| + |\mathcal{J}_2(Q(SSingO_2))| = 2 + 2 = 4$
$|Q(SSO_2)| = |\mathcal{J}_2(Q(SSO_2))| = 3$

IV. MAIN RESULTS IN $Q(SSingO_n)$ AND $Q(SSO_n)$

From the last tables, we developed the following sequence of enumerations of $Q(SSingO_n)$ and $Q(SSO_n)$ for small values of n and this happens for all values of $n \geq 1$. We should also note that all the elements in red are symmetric inverse semigroup elements of order-preserving. The triangle array display below, summarized the findings of this paper. We also developed theorems that determined the total number of quasi-idempotent elements of order-preserving.

Table 4.1: Quasi-idempotents in order-preserving signed full transformation Semigroup, $|Q(SSing_n)|$.

n	1	2	3	4
$Q(SSingO_n)$	0	4	54	401
$Q(SSO_n)$	1	3	7	15

Lemma 4:1
Let $\alpha \in ST_n$ be a signed full transformation. If ($\alpha \in SS_n$, $-\left\{\begin{pmatrix} 1, & 2, & \cdots & n \\ 1, & 2, & \cdots & n \end{pmatrix}\right\}$). Then α is quasi-idempotent element.

Proof.
Let $\alpha \in ST_n$ be self-inverse element (that is $\alpha^2 = i$), where i is an identity elements. Thus,
We have,

$\alpha^{2+0} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ \pm t_1 & \pm t_2 & \cdots & \pm t_n \end{pmatrix} \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ \pm t_1 & \pm t_2 & \cdots & \pm t_n \end{pmatrix} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} = i$ and

$\alpha^{n+2} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ \pm t_1 & \pm t_2 & \cdots & \pm t_n \end{pmatrix}^{n+2} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} = i$,

$\alpha^{n+4} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ \pm t_1 & \pm t_2 & \cdots & \pm t_n \end{pmatrix}^{n+2} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} = i$. Its follow that,

$\alpha^{n+r} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ \pm t_1 & \pm t_2 & \cdots & \pm t_n \end{pmatrix}^{n+r} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} = i$, for $n = 2,4, \ldots$ and $r = 0,2,4 \ldots$ Since $\alpha^2 = i = \alpha^4$, hence Self-inverse elements are quasi - idempotents.

Theorem 4.1
Let $S = Q(SSO_n)$, if $n \leq 4$, then $|S| = 2^n - 1$.
Proof.
Let $\mathrm{Z_n} = \{1,2, \ldots, \mathrm{n}\}$ and $\mathrm{Z_n}^* = \{\pm 1, \pm 2, \ldots, \pm \mathrm{n}\}$ such that $\mathrm{dom}(\alpha) \subseteq \mathrm{Z_n} \longrightarrow \mathrm{im}(\subset)\mathrm{Z_n}^*$, $\mathrm{dom}(\alpha) = \mathrm{Z_n}$, $\mathrm{im}(\alpha) \subset \mathrm{Z_n}^*$ and then

any $\alpha \in SSO_n$ in $SS_n \subseteq ST_n$ is such that $\alpha^2 = i$ and $|\mathrm{im}(\alpha)| = n$ for SS_n where $-i \neq i$. We observed that from the sequence generated, the order of $Q(SSO_n)$ is one less than the power set. Hence by the result for $n = 1,2, \cdots,$ $Q(SSO_n) = 2^n - 1$, as required.

Theorem 4.2
Let $S = Q(SSingO_n)$, if $n \leq 3$, then $|S| = n^n(n-1)$.
Proof.
Let $\alpha \in Q(SSing_n)$, such that $\alpha: \mathrm{dom}(\alpha) \longrightarrow \mathrm{im}(\alpha)$. The number of each image, that an element α can have is $n-1$ ($|\mathrm{im}(\alpha)| = n-1$), from theorem 4.1 $\alpha \in Q(SSO_n)$, $|\mathrm{im}(\alpha)| = n$ when $-i \neq i$, then there are n^nelements for $i = 1,2,3, \ldots, n$. The nature of the $\mathrm{im}(\alpha)$ is that $\mathrm{im}(\alpha) \subseteq {Z_n}^*$ Hence the product rule implies $|S| = n^n(n-1)$ elements for $n \leq 3$.

Theorem 4.3
Let $S = Q(SSingO_n)$, if $n \geq 4$, then $|S| = n^n + 5^{n-1} + n(n+1)$.
Proof.
By applying the principle of Mathematical induction, the results is true for $n = 4$. Suppose $n = k$ (by induction hypothesis), the result is true for all k. That is; $k^k + 5^{k-1} + k(k+1)$. We also, assume $n = k + 1$ the we have $(k+1)^{k+1} + 5^k + (k+1)(k+2)$.
Thus, by induction the result hold for all $n \in \mathbb{Z}^+$. Hence, the proof is complete.

V. CONCLUSION AND RECOMMENDATION

5.1 Conclusion

The study focuses on the combinatorial and algebraic properties of quasi-idempotent elements on signed singular mapping of full transformation semigroup, It also gives formulae for these quantities. This paper gives new results on the number of elements in signed singular mapping, together with their accompanying proofs, and unifies prior results in combinatorials.
The findings indicated that quasi-idempotent elements in signed transformation semigroup is a topic for investigation in the theory of transformation semigroup, and many work in this field is possible.

5.2 Recommendation

Signed transformation semigroup is a new area research, we suggest expanding this kind of research to signed transformation order-decreasing.

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