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Estimation of Cobb-Douglas Production Function^{0}s Parameters

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ABSTRACT: Economic growth and development is one of the major goals of any economy. Cobb-Douglas production function is an important model for analysis of productivity and progress. Its parameters measure the individual factor and overall performance of an economy. The sum of the elasticities of the factors are needed to be greater than one for an economy to grow and hence develop. This essay shows how the parameters of the Cobb-Douglas production function are estimated using Ordinary Least Squares (OLS) method for an economy or industry of interest. It also shows how test of hypothesis can be conducted such as whether the sum of labour and capital inputs elasticities are greater than one or not.

KEY WORDS: OLS, Cobb-Douglas, Production Function, Parameter, Return to Scale.

I. INTRODUCTION

The major aim of a rm is to maximize output, minimize cost and hence maximize pro t. There is trade-o between the two. As a result, rms are faced with the problem of having optimum combination of inputs which maximize their level of output. Conversions of inputs into a nal products is called production. The algebraic expression between speci ed level of inputs and maximum level of output a rm can obtain from those inputs is known as Production function. In general, the production function can be written algebraically as:

$$X = f(Y); \qquad (1)$$

where X is the amount of output produced, Y is the vector of factors employed to produce X level of output and f is a functional relationship. The inputs might be more than two, but in real world capital and labour are the most important factors and substituting them with each other is the most problematic (Rasmussen, 2013).

The above production function describes the optimum combination of factors Y to get maximum products X. Of course, ine ciency could result to lower output (Besanko and Braeutigam, 2011).

It cost rms a huge amount of money to purchase and install sophisticated machines. Therefore, before embarking on such decision, it might be necessary to know the rate at which it will substitute one factor (say labor) with another (say capital) and keeping the level of its nal products unchanged.

The ease with which rm can substitute one factor with another is referred to as MRTS (Besanko and Braeutigam, 2011).

Isoquant is a curve that represents production function diagrammatically. It shows the ability of a rm to substitute one factor with another while maintaining the same level of output. Isoquant exhibits di erent shapes depending on the rm's technological level.

Cobb-Douglas (C-D) production function is a type of production function which assumes the use of only two factors to produce given level of outputs. The original and general deterministic C-D production function is in the form:

$$X_i = AY_{i1}^{\alpha}Y_{i2}^{\beta}, \qquad (2)$$

where X_i represents the maximum level of output, A represents the total factor productivity, Y_1 represents input 1, Y_2 represents inputs 2, and represent the productivity of factor Y_1 and factor Y_2 respectively. andalso shows the partial elasticity of their respective factors. This production function was originally appraised in 1927 in an attempt to explore the behaviour of output in response to di erent level of factor employed (Biddle, 2012).

Estimating the parameters (A, and) is the one of the major concern of a rm (or a nation, as the case may be) because they show the level of productivity and the return to scale of a rm or a

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nation. The C-D production function is nonlinear in nature but it can be linearized by taking the log of both sides and use OLS to estimate its parameters.

There are other methods of estimating the parameters of the model, in which the industry (or sector) in question determines the one to use which is based on accepting alternative hypothesis (H_1) that there is signi cance di erence between the various methods. These methods include input proxy estimator, Instrumental Variable estimator b_{IV} , GMM estimator (Levinsohn and Petrin, 2003). Hossain et al. (2012) used Newton-Raphson Method for the estimation because he assumes an additive error term instead multiplicative, in that his model is no more a linear model. Goldfeld (1972) estimated the parameters using simultaneous multiplicative and additive error terms. Also, Coelli et al. (2005) suggested the use of Maximum Likelihood estimator.

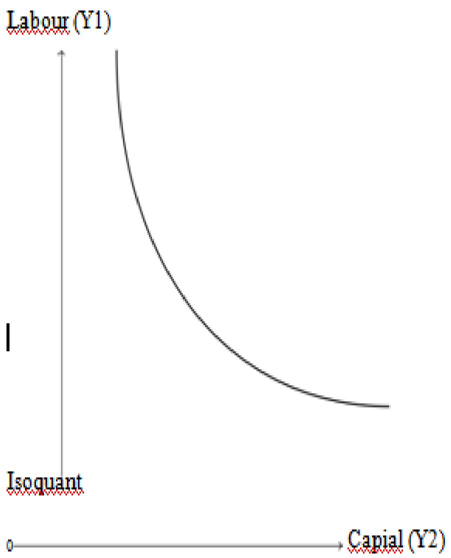
The aim of this essay is to show the procedure for estimating these parameters using OLS.

II. BASIC MODELS

2.1 Production Function

Production function is a mathematical relationship between optimum output and vector of inputs:

Figure 1: CD PF Isoquant



$$X = f(Y) \qquad (3)$$

where X is a scalar vector of output, f is the functional relation and Y is the vector of inputs. X is the maximum amount of output that can be produced given the quantity of inputs employed. These inputs include capital, labour, raw materials, management, etc. Usually, two inputs (labour and capital) are represented because substituting them is the most problematic in the production process, therefore, it takes the form: $X = f(Y_1; Y_2)$. Labour and capital can be substituted with each other. the rate at which they are substituted is called Marginal Rate Technical Substitution (Rasmussen, 2013).

2.1.1 Isoquant

An isoquant is a graphical representation of a production function. It shows the ability of a rm to substitute one input with another while maintaining the same level of output. it exhibits di erent shapes depending on the rm's (or nation's) ability. It can be:

linear: $X = Y_1 + Y_2$,

L-shape: $X = \min(Y_1 + Y_2)$,

a curve: $X = AY_1 Y_2$.

The Isoquant in Figure 1 is for C-D Production Function which is curvature $X = AY_1 Y_2$, this shows that labour and capital can be substituted almost perfectly. A shift in the Isoquant either up or down indicates the increases or decreases in Total Factor Productivity respectively (Besanko and Braeutigam, 2011).

2.1.2 Elasticity of Substitution

The mathematical aspect that aids in expressing the rate at which one factor can be substituted with another by moving on the same isoquant is elasticity of substitution ( );

$$\sigma = \frac{\%\Delta(Y_1/Y_2)}{\%\Delta MRTS_{Y_1Y_2}}$$

Movement along an isoquant from one point to another is also referred to as elasticity of substitution. The value of elasticity of substitution ranges from 0 to 1. It is in nity when the production function is linear in nature and the its parameters are xed. Its 0 when it exhibits xed production function, this type of production function is usually known as Leontie production function (Besanko and Braeutigam, 2011).

2.2 Cobb-Douglas Production Function

"The Cobb-douglas Production Function is still today the most ubiquitous form in theoretical and empirical analyses of growth and productivity. The estimation of the parameters of the aggregate production function is central to much of today's work on growth, technological change, productivity and labour. Empirical estimates of aggregate production functions are tool of analysis essential in macroeconomics, and important theoretical

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consructs, such as potential output, technical change, or the demand for labour are based on them" (Felipe and Adams, 2005).
Production Functions are used at micro and macro level of an economy. The original version of Cobb-Douglas production function is given as:

$$Y_i = \beta_0 L_i^{\beta_1} K_i^{1-\beta_1} \quad (4)$$

This was estimated using the data of US manufacturing industry from 1889-1922 which shows a constant returns to scale (+ 1 = 1) with the use of only two factors, Labour and Capital. Cobb-Douglas Production Function was not the rst production function in Economics. Scholars before like David Ricardo, Thomas Malthus and others have implicitly used one kind of production function or the other before the 20th century (Humphrey, 1997).
Goldar et al. (2013), equation (4) seems to be nonlinear, but it can be linearized by log-transformation as in equation

(5) for the estimation purpose and/or when the errors are heteroskedastic . log $_o$ is the total factor productivity, $_1$ and $_2$ (estimated) are the productivity levels for the inputs. $\log Y_1$ and $\log Y_2$ are the observed inputs. C-D production function is one of the most important concept in Economics because of the role it plays in estimating the growth of an industry or a nation. its parameters show the contribution of each factor to the total production. export and hence favourable balance of trade and consequent economic growth. Let be the percentage change in input and be the percentage change in output.

$$\log Y_i = \log \beta_0 + \beta_1 \log L_i + (1-\beta_1)\log K_i \quad (5)$$

That is both of the inputs are increased by the same factor. Clearly, if:

1. + > 1 denotes Increasing RTS
2. + < 1 denotes Decreasing RTS
3. + = 1 denotes Constant RTS

(Besanko and Braeutigam, 2011).

The major aim of this essay is to show how these parameters are estimated.
Hypothesis such as

H_o : + =6 1 is tested against H_1 : + = 1:

III. THE MODELS

3.1 Stochastic Cobb-Douglas Production Function

In practice, the deterministic model described above will not generate good and efficient estimates of the parameters because it does not give room for error term. This led to the development of the stochastic model which accomodates the existance of multiplicative error term as: Xi = AY α i1Y β i2 e εi (7) which can be linearized by means of log-transformation as: log Xi = logA + α logYi1 + β logYi2 + εi (8) The model has now bocome multiple linear regression model which can be estimated by OLS (Coelli et al., 2005) OLS assumes that the regressors are not correlated with the error term. C-D Production Function model assumes that only two inputs (labour and capital) are used to produce the given level of output, other factors exist and known by the firm like management, government policy, etc but they can not observed. These factors are termed as Total Factor Productivity represented by A in equation (9). Hence OLS can be used for the estimation. Even though (Arguirregabiria, 2009) used the model without an intercept (A, the total factor productivity) and argued 3 that an instrument such as price of factor can be used and applied Instrumental variable IV estimator for the parameters estimation.

Ordinary Least Squares (OLS) OLS is well known due to its simplicity. It minimises the sum of squared residuals to estimate the parameters of a regression model. The general form of OLS is given as:
Xi = β0 + β1Yi1 + β2Yi2 + · · · + βkYik + εi , (9)
where Yi is dependent variable, βo . . . βk are the estimated unobserved parameters, X1 . . . Xn are the observed regressors and i is the unobserved disturbance shock. The model can be written more compactly in matrix form as:
X = Y β + □ (10)
X = N × 1 vector of dependent variable, Y = N × N regressors, β = K × 1 vector of unobserved but estimated parameters and i is the disturbance shock. The main target here is to estimate the parameters and conduct a test of hypothesis. For OLS to be unbiased, one needs
E[b|X] = β
For simplicity, let equation (8) be written as:
Xi = βo + β1Yi1 + β2Yi2 + ε, (11)

where:
• X = logXi the log of Output/GDP,
• βo = logA, the logarithm of Total Factor Productivity,
• β1 = elasticity of labour, • β2 = elasticity of capital,
• Y1 = logYi , the log of hours of labour employed,
• Y2 = logYi , the log of capital/machines hours used
• ε = the unobserved shock (error term).

To estimate the parameters of interest using OLS method, the sum of squares residual (Pn i=1 2 i) is minimized, letting
Pn i=1 □ 2 i = S(β) = □ 0 □ S(β) = (X − Y β) 0 (X − Y β) (12)

Differentiating equation (12) partially with respect to β (i.e. taking a directional derivative along the vector β), setting it equals to zero and finding the solution of the equation give the estimate of the parameter vector. Here the vector

$$\frac{\partial(S\beta)}{\partial\hat{\beta}} = \begin{pmatrix} \frac{\partial(S\beta)}{\partial\beta_1} \\ \frac{\partial(S\beta)}{\partial\beta_2} \\ \frac{\partial(S\beta)}{\partial\beta_3} \end{pmatrix}$$

is the gradient of $S\beta$ along the vector β. Hence

$$\frac{\partial(S\beta)}{\partial\hat{\beta}} = -2Y'X + 2Y'Y\hat{\beta} = 0 \qquad (13)$$

The solution to equation (13) is the estimate of the OLS.

Therefore,

$$-2Y'X = -2Y'Y\hat{\beta} \qquad (14)$$

and with the assumption of full rank, we have:

$$\hat{\beta} = (Y'Y)^{-1}Y'X. \qquad (15)$$

Equation (14) is called normal equations. To ensure that βˆ corresponds to the minimum, second order condition is checked, which is differentiating equation (13) with respect to β. This gives 3×3 Hessian matrix of the second order derivatives as:

$$H(\hat{\beta}) = \frac{\partial^2 S(\beta)}{\partial\hat{\beta}'\partial\hat{\beta}} = \begin{pmatrix} \frac{\partial^2(S\beta)}{\partial\beta_0^2} & \frac{\partial^2(S\beta)}{\partial\beta_1\partial\beta_0} & \frac{\partial^2(S\beta)}{\partial\beta_2\partial\beta_0} \\ \frac{\partial^2(S\beta)}{\partial\beta_0\partial\beta_1} & \frac{\partial^2(S\beta)}{\partial\beta_1^2} & \frac{\partial^2(S\beta)}{\partial\beta_2\partial\beta_1} \\ \frac{\partial^2(S\beta)}{\partial\beta_0\partial\beta_2} & \frac{\partial^2(S\beta)}{\partial\beta_1\partial\beta_2} & \frac{\partial^2(S\beta)}{\partial\beta_2^2} \end{pmatrix}$$

$$\frac{\partial^2 S(\beta)}{\partial\hat{\beta}^2} = 2Y'Y \qquad (16)$$

which is positive, so βˆ corresponds to the minimum (Verbeek, 2004).

3.2.1 Assumptions of OLS The main target is to show how how the parameters of C-D PF are estimated, but the following assumptions should be considered (Greene, 2012):
1. The model should be linear in parameters, this means that β1 and β2 are linear in nature, that is X = Y β +ε
2. X is a full rank matrix, that is the columns of Xi are not linearly correlated with one another. 4
3. The regressors are exogenously determined, that is E(ε|Y) = 0 and hence E(ε) = 0. This assumptions means that the regressors are uncorrelated with the error term, that is ε and Y are independent
4. Spherical Disturbance: This assumption captures two important issues about the error term, ε. First is that the errors should be homoskedastic, this means variance of the errors should be constant (σ 2 IN) and the second ensures the absence of autocorrelation. Spherical disturbance assumption can be summarized as:
Var [□ i |Y] = σ 2 IN , cov [□ i , □ j |Y] = 0,

Where IN is the identity matrix.
Autocorrelation means that the error is correlated with its past values. 5. Normality:

(i |Y) ∼ N(0, σ2 IN).

3.2.2 Properties of OLS Two important properties of OLS regarding its parameter and variance are: 1. The E(βˆ) = β for OLS estimator to be an unbiased estimator (Verbeek, 2004) Hence, βˆ is an unbiased estimator of β, which means that the average value of βˆ is β. Remark 1: If E(βˆ) 6= β, that is when E(ε|Y) 6= 0 (a situation where ε and Y are not independent), then OLS estimator produces a biased result as E(βˆ) = β + (Y 0Y) −1Y 0E(ε) 6= β, since E(ε) 6= 0. In this situation, another estimator such as Instrumental Variable estimator bIV will be used where another independent variable which is

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uncorrelated with the error term and correlated with the regressor that correlates with error term is included in the model as an Instrument, thereafter the parameters is then estimated using the Instrumental variable estimator bIV . In this essay, it is assumes that E(βˆ|Y) = β that is E(ε|Y) = 0. Hence OLS is still applicable.

2. *The* $Var(b|Y) = \sigma^2(Y'Y)^{-1}$ *(Verbeek, 2004).*

Remark 2: If E(εε0 |Y) 6= σ 2 IN , but rather E(εε0 |Y) = σ 2Ω, where Ω is N × N matrix. This might result in either or both of the following two issues:

• The diagonal elements of Ω are not identical (that is the errors are heteroskedastic), nonconstant variance or they are identical but,

• The opposite diagonal elements are non-zero that is E(εiεj) 6= 0, then the errors are serially correlated. In any of the above two cases, generalzed Least Squares estimator is used because OLS will results in biased result. A test for heteroskedasticity and that of serial correlation can be conducted to ensure the presence or absence of heteroskedasticity or serial correlation. But in our OLS model, we assume that var(βˆ|Y) = σ 2 IN and E(ε1εj |Y) = 0, which implies the applicability of OLS for our model estimation.

3.2.3 The Parameters Estimation Issues

In the C-D Production Function, some factors are known but can not be observed and hence not included in the model. Also, the regressors are sometimes correlated. These have raised some estimation issues:

1. Multicollinearity.

Multicollinearity happens in a situation where at least two of the regressors are signi cantly correlated. In equation (7), Y_1 and Y_2 might be correlated. This is the case because production can not take place without the combination of both Labour and Capital. They will not completely substitute each other, elasticity of substitution in C-D is 1(= 1). The model gives the overall impact of the regressors on the regressant, sometimes they can not give the appropriate individual impacts. If one the regressors is dropped and the other one is signi cantly a ected, then there is existance of multicollinearity. This might happens in C-D PF, because it needs the contribution of both capital and labour. OLS can still be used here since the essence is just to estimate the parameters and see the elasticity of each when both factors are combined together. Also, in the 21st century due to advent of modern technology, production takes place with the contribution of capital to be more than that of labour. Verbeek (2004), argued that two variables can be correlated in a model so long as the correlation is small otherwise it results to inefficency in estimation.

2. Simultaneity.

OLS assumes that the regressors are exogeneously determined. Amount of labour employed normally depends on the required level of output to produce, wage rate, total income and even its productivity. This leads to simulataneity problem. But we should only compute the extent of the dependency not preventing us from using OLS (Griliches and Ringstad, 1971).

3. Total factor Productivity.

In C-D Production Function, some factors make significant contributions to the output level and measure GDP growth of an economy but are unobserved. Such factors are referred to as Total Factor Productivity. A in equation (9) captures all those factors which include managerials skills, technical know-how, technological progress, quality of labour and quality of land. For instance in the period between 1947 and 1973 in United Kingdom, the contributions of capital to GDP growth was 47%, labour 1% and Total Factor Productivity 52% (Easterly and Levine, 2001). This argument suggests that C-D Production Function parameters can be estimated using OLS since the regressors are not correlated with the error term, unobserved factors are taken care of by A, the Total Factor Productivty.

3.2.4 The Hypothesis

In equation (7), α and β are the elasticities. They show the contributions of capital and labour respectively to total output (GDP). In this regard, null hypothesis is tested to find out whether their sum is added up to one for constant returns to scale or not. The hypothesis is thus:

Ho : α + β = 1 against H1 : α + β 6= 1

Another hypothesis is also tested to see whether an individual factor(capital or labour) contributes significantly to the total output as:

Ho : α = 0 against H1 : α 6= 0 and Ho : β = 0 against H1 : β 6= 0 (Greene, 2012).

In C-D Production Function, hypothesis can also be tested as: Ho : α + β = 0 against H1 : α + β 6= 0 This will prove the overall influence of the factors on the output (Gujarati, 2003).

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3.2.5 Coefficient of Determination R2

The quantity R2 measures the goodness of fit. It measures the extent to which the explanatory variables explain the dependent variable. The value of R2 is: 0 ≤ R2 ≤ 1. R2 close to 0 means that the model explains nothing while R2 close to 1 shows that the line fits the data very well. Therefore, higher value of R2 is required for the regressors to be used as explanatory variables in the regression, it is given as:

$$R^2 = 1 - \frac{\sum \hat{\epsilon}_i^2}{\sum (X_i - X)^2}$$

IV. OUTLOOK

4.1 Data

To estimate the parameters of interest, data is needed which can be cross sectional, time series or longitudinal. Each type of data can be used for the estimation depending upon the objectives of the study. For instance, (Levinsohn and Petrin, 2003) used longitudinal data from 1979-1986 involving four industries (Food products, metals, textiles and wood products) and estimated the parameters. Kumbhakar (2012), used cross sectional data using a sample size of 3249 firms (tree harvesters) in Norway in 2003. Also, Prajneshu (2008), used time series data of Punjab from 1971-2000 for wheat industry.

Based on the above arguments, data is needed for the estimation. But since the objective of this essay is to show the procedure of estimation by OLS, the use of cross 6 sectional data is recommended in this case because longitudinal and time series data may exhibit an autocorrelation issue. The data can be at microeconomic which invloves specific firms or industries or at macroeconomic level involving a particular nation or region as a whole.

Nowadays, several type of statistical softwares and packages like SPSS, EViews, Stata, R, et cetra; are available for parameters estimations and other statistical anlysis, which can also be used for the estimation of this parameters.

V. CONCLUSION

This essay shows how the parameters of stochastic Cobb-Douglas production function with multiplicative error term is estimated using the method of OLS. It is suggested that the data to be used for the estimation should be cross sectional because available literature shows that OLS is not appropriate for time series or longitudinal data.

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