International Journal of Advances in Engineering and Management (IJAEM)
Volume 4, Issue 10 Oct. 2022, pp: 1295-1298 www.ijaem.net ISSN: 2395-5252

Fekete Szego Coefficient Inequality for a Subclass of Analytic Functions

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Date of Submission: 15-10-2022 Date of Acceptance: 31-10-2022

ABSTRACT –In the presentpaper, we solve coefficient inequality proved by Fekete and Szegö[5] in 1933 by using the analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ for a new class.

2010 Mathematics Subject Classification: 30C45, 30C50.

Keywords - Principle of subordination, Fekete – Szegö Inequality and concept of Bounded analytic functions.

I. INTRODUCTION

We are dealing with geometric function theory, it is that branch of complex analysis which deals with the analytic functions geometrically. The pillar of this theory is Riemann Mapping Theorem which was proved in 19th century. It initiated its roots in the work of great mathematician Koebe [19] in 1907, who stated that "An analytic function which is univalent has properties of conformal mapping i.e. angle preserving property". From this theorem, Bieberbach conjecture was proved. This was given by L. Bieberbach[2] in 1916 but proved finally by Louis De Branges [3] in 1985 and while tackling with this conjecture, an equality arises, which is called FeketeSzegö Inequality given by Fekete and Szegö [5].

The inequality which is for the function $f(z) \in A$ and based on Bieberbach conjecture, is named as FeketeSzegö Inequality, which states that if f(z) is a function of type

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which is univalent in E, then

$$|a_3 - \mu a_2^2| \le \begin{cases} 3 - 4\mu \text{ if } \mu \le 0 \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) \text{ if } 0 \le \mu \le 1 \\ 4\mu - 3 \text{ if } \mu \ge 1 \end{cases}$$

This is an inequality which is related to univalent analytic functions [8],[16] and gives the necessary condition to map the unit disk of a complex plane injectively to the complex plane. It gives the relation between second and third coefficient of univalent analytic function.

In order to prove our result, let us explain some classes and some basic results related to our work:-

Aconsists all those functions fwhich are analytic in open unit disc $E = \{ z \in C:|z|<1\}$ and are of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, with normalization conditions $f(0) = 0, f'(0) = 1$.

S be the family of functions fwhich are univalent in the open disk $\{ z \in C:|z|<1\}$ with conditions $f(0) = 0$, $f'(0) = 1$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

$S^*(\phi)$ be the class of functions in $f \in S$, for which $\frac{z f'(z)}{f(z)} \prec \phi(z)$, given by Ma and Minda [10].

t(z) be a family of analytic functions in the open unit disk E, having functions of the form $t(z) = \sum_{n=1}^{\infty} c_n z^n$, it is a class of bounded analytic function denoted by U, if the conditions $t(0) = 0$ and $|t(z)| < 1$ hold. The necessary conditions for any function to be bounded analytic function are $|c_1| \le 1$, $|c_2| \le 1-|c_1|^2$; which were given by Miller et. al. [11],

Let u (z) and v(z) are two analytic functions in E. If there existsa Schwarzian function F(z) (analytic in E) in such a way that $|F(z)| < 1$, $F(0) = 0$ and $u(z) = v(F(z))$; $z \in E$ then the function u(z) is subordinate to v(z) written as $u(z) \prec v(z)$ and this concept (called subordination) was given by Lindelof [9].

We introduce a new class $Q^{\alpha}(f, A, B, \delta)$ offunctions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; defined as

$$\frac{f(z) - f(\alpha z)}{(1-\alpha)z} = \left(\frac{1 + Aw(z)}{1 + Bw(z)}\right)^{\delta} ; z \in E. \ldots (1.1)$$

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II. MAIN RESULTS

THEOREM1:- Let $f(z) \in Q^{\alpha}(f, A, B, \delta)$ and $\phi(z) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta}$;w(z) is a Schwarzian function , then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{\delta B(B-A)}{\alpha^2-\alpha+1} - \frac{\mu(A-B)^2\delta^2}{(1+\alpha)^2} ; \mu \le \frac{(\delta B+1)(1+\alpha)^2}{\delta^2(B-A)(\alpha^2-\alpha+1)} ; \\ \frac{A-B}{\alpha^2-\alpha+1} ; \frac{(\delta B+1)(1+\alpha)^2}{\delta^2(B-A)(\alpha^2-\alpha+1)} \le \mu \le \frac{(\delta B-1)(1+\alpha)^2}{\delta^2(B-A)(\alpha^2-\alpha+1)} ; \\ \frac{\mu(A-B)^2\delta^2}{(1+\alpha)^2} + \frac{\delta B(A-B)}{\alpha^2-\alpha+1} ; \mu \ge \frac{(\delta B-1)(1+\alpha)^2}{\delta^2(B-A)(\alpha^2-\alpha+1)} . \end{cases}$$

PROOF: - By definition of $Q^{\alpha}(f, A, B, \delta)$, f(z)), given by (1.1)
and using $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \dots$, $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$
$f(\alpha z) = z + a_2(\alpha z) + a_3(\alpha z)^2 + a_4(\alpha z)^3 + \dots$, we get
we get

$$1 + a_2 z(1+\alpha) + a_3(\alpha^2 - \alpha + 1)z^2 + \dots$$
$$= 1 + (A-B)\delta c_1 z + (A-B)(c_2 - B\delta c_1^2)z^2 + \dots$$

Comparing like coefficients, one can easily obtain

$$a_2 = \frac{(A-B)\delta c_1}{1+\alpha} \text{ and } a_3 = \frac{(A-B)c_2}{(\alpha^2-\alpha+1)} + \frac{B(B-A)\delta}{(\alpha)^2-\alpha+1} c_1^2$$

Using these values of a_2 and a_3 , one can construct

$$a_3 - \mu a_2^2 = \frac{(A-B)c_2}{(\alpha^2-\alpha+1)} + \left(\frac{B(B-A)\delta}{\alpha^2-\alpha+1} - \frac{((A-B)\delta)^2\mu}{(1+\alpha)^2}\right) c_1^2$$

After applying mode on both sides, we get

$$|a_3 - \mu a_2^2| \le \left(\frac{A-B}{(\alpha^2-\alpha+1)}\right)|c_2| + \left|\frac{B(B-A)\delta}{\alpha^2-\alpha+1} - \frac{((A-B)\delta)^2\mu}{(1+\alpha)^2}\right| |c_1|^2$$

Using $|c_2| \le 1 - |c_1|^2$, we get

$$|a_3 - \mu a_2^2| \le \frac{A-B}{(\alpha^2-\alpha+1)} + \left\{\left|\frac{B(B-A)\delta}{\alpha^2-\alpha+1} - \frac{((A-B)\delta)^2\mu}{(1+\alpha)^2}\right| - \frac{A-B}{(\alpha^2-\alpha+1)}\right\}|c_1|^2$$

Case 1:- If $\mu \le \frac{(1+\alpha)^2 B}{\delta(B-A)(\alpha^2-\alpha+1)}$

Then, $|a_3 - \mu a_2^2| \le \frac{A-B}{(\alpha^2-\alpha+1)} + \left\{\frac{(B-A)(B\delta+1)}{\alpha^2-\alpha+1} - \frac{((A-B)\delta)^2\mu}{(1+\alpha)^2}\right\}|c_1|^2$

Subcase – 1 (a):- When $\mu \le \frac{(\delta B+1)(1+\alpha)^2}{\delta^2(B-A)(\alpha^2-\alpha+1)}$

By using $|c_1| \le 1$, we get

$|a_3 - \mu a_2^2| \le \frac{\delta B(B-A)}{\alpha^2-\alpha+1} - \frac{\mu(A-B)^2\delta^2}{(1+\alpha)^2}$ ---------- (1.2)

Subcase – 1 (b):- When $\mu \ge \frac{(\delta B+1)(1+\alpha)^2}{\delta^2(B-A)(\alpha^2-\alpha+1)}$

Then, $|a_3 - \mu a_2^2| \le \frac{A-B}{(\alpha^2-\alpha+1)}$ -------- (1.3)

Case – 2:- If $\mu \ge \frac{B(1+\alpha)^2}{\delta(B-A)(\alpha^2-\alpha+1)}$

Then, $|a_3 - \mu a_2^2| \le \frac{A-B}{(\alpha^2-\alpha+1)} + \left\{\frac{((A-B)\delta)^2\mu}{(1+\alpha)^2} - \frac{(A-B)(1-B\delta)}{\alpha^2-\alpha+1}\right\}|c_1|^2$

Subcase-2 (a):- When $\mu \ge \frac{(\delta B-1)(1+\alpha)^2}{\delta^2(B-A)(\alpha^2-\alpha+1)}$

Then, $|a_3 - \mu a_2^2| \le \frac{\mu(A-B)^2\delta^2}{(1+\alpha)^2} + \frac{\delta B(A-B)}{\alpha^2-\alpha+1}$ ----- (1.4)

Subcase – 2 (b):- When $\mu \le \frac{(\delta B-1)(1+\alpha)^2}{\delta^2(B-A)(\alpha^2-\alpha+1)}$

Then, $|a_3 - \mu a_2^2| \le \frac{A-B}{(\alpha^2-\alpha+1)}$ -------- (1.5)

Combining (1.2), (1.3), (1.4) and (1.5) , we get the required result.

Corollary 2:-Putting $\alpha = 0$, the result becomes

$$|a_3 - \mu a_2^2| \le \begin{cases} 2 - 4\mu; \ \mu \le 0; \\ 2; \ 0 \le \mu \le 1; \\ 4\mu - 2; \ \mu \ge 1. \end{cases}$$

which is the required result given by Gurmeet Singh , Misha Rani [18].

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