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Fractional Nonlinear Partial Differential Accretive Operators in a Closed, Bounded, and Continuous Domains.

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ABSTRACT
This research analyzes the properties of a nonlinear operator A acting on functions u belonging to the domain $D(A)$, which is a subset of a Banach space X (e.g.,$L^2([0,1])$). The operator A takes the form in (1.1).
It is established that under certain assumptions on the operator A and the function F, the following properties hold: **A is accretive and M-accretive** in the Banach space X, **A has a unique solution** for every $f \in X$ and $\lambda > 0$, **A is coercive and hemicontinuous** and **A is stable**, meaning small perturbations in the initial conditions lead to small changes in the solution.
The proof of these properties relies on the theory of accretive operators, the Browder-Minty theorem, and the analysis of the fractional derivative and nonlinear terms. The stability analysis assumes that the nonlinear term F satisfies a Lipschitz condition and that the gradient term can be controlled by the function itself. These results establish the well-posedness and stability of the operator A, which is crucial for the analysis and numerical approximation of problems involving fractional differential equations with nonlinear terms.
Keyword: Fractional Nonlinear Partial Differential, Accretive Operators, Stability, Coercivity and Hemi continuity.

I. INTRODUCTION

Combining the intricacies of fractional calculus, nonlinear dynamics, and accretive operator theory, fractional nonlinear partial differential accretive operators (FNPDAOs) in closed, bounded, and continuous domains provide a complicated extension of classical partial differential equations. Because of its capacity to simulate intricate phenomena including memory effects, non-local interactions, and dissipative behaviours, this topic has attracted a lot of interest [1].

The characteristics of operators defined on function spaces are essential to comprehending the existence and uniqueness of solutions to distinct mathematical problems in the fields of functional analysis and partial differential equations.

$$Au(x) = \frac{\partial^\alpha}{\partial t^\alpha} u(x) + F(x, u(x), \nabla u(x)) \qquad 1$$

is one such operator. Here, α represents a fractional derivative, and F is a nonlinear function that relies on the function u, the spatial variable x, and its gradient ∇u. Establishing the mathematical foundations required for the application of variational techniques and fixed-point theorems in the setting of nonlinear partial differential equations requires an investigation of this operator.
Some important properties of the operator A that we will investigate are accretive, existence and unique, coercivity and hemicontinuity.
- Coercivity; ensures that the operator provides a lower bound on the inner product of the operator applied to a function with that function itself, which is crucial for demonstrating the boundedness and stability of solutions [2].
- Hemicontinuity; is a property that guarantees the continuity of the operator in a weak sense, allowing us to infer the convergence of sequences of functions in the context of the operator.

Establishing the existence and uniqueness of solutions for such operators is crucial for their theoretical understanding and practical application. This introduction outlines an approach to prove the uniqueness of solutions using the Banach fixed-point theorem, also known as the contraction mapping principle. This approach requires careful analysis of the fractional derivative term and the nonlinear term $F(x, u(x), \nabla u(x))$. The fractional derivative is often handled using properties of fractional calculus, while the nonlinear term typically requires Lipschitz continuity assumptions.

This work is in line with Echude et al (2017, 2024) were 'a non-linear parabolic partial differential equation in a closed, bounded and continuous domain' is now extended to 'fractional nonlinear partial differential accretive operators in closed, bounded and continuous domains'. They investigated on a class of non-linear parabolic partial differential equation in a closed, bounded and continuous domain which is done by converting such a non-linear parabolic partial differential equation to an abstract Cauchy problem, the operator was shown to be m-accretive and therefore established that the partial differential equation has a solution by the fundamental results of accretive operators [3,4].

The use of functional analysis techniques, particularly the theory of monotone operators and accretive operators, plays a crucial role in this analysis [5,6]. These tools allow for the treatment of a wide class of nonlinear problems and provide a robust framework for establishing well-posedness [7].

The nonlinear aspect of FNPDAOs introduces additional complexity, capturing phenomena such as self-organization, pattern formation, and chaotic behavior that are prevalent in many real-world systems [8]. Nonlinearity often leads to multiple equilibria, bifurcations, and sensitive dependence on initial conditions, making the analysis and numerical treatment of these operators particularly challenging [9].

The accretive property of these operators is crucial in ensuring the well-posedness of associated initial and boundary value problems [10]. Accretive operators generalize the concept of monotonicity and play a vital role in the study of dissipative systems, providing a framework for establishing existence, uniqueness, and stability of solutions [11].

The study of Fractional Nonlinear Partial Differential Accretive Operators in a closed, bounded, and continuous domains represents a confluence of several advanced mathematical concepts. It offers both theoretical challenges and practical applications, making it a rich area for ongoing research and interdisciplinary collaboration.

II. MATHEMATICAL FORMULATION

(Theorem 1)

Let's denote the operator by A. A operates on functions u belonging to the domain D(A) which is a subset of a Banach space X (e.g.,$L^2([0,1])$). A takes the form:

$$Au(x) = \frac{\partial^\alpha}{\partial t^\alpha} u(x) + F(x, u(x), \nabla u(x)) \qquad 1.1$$

where:

- $\frac{\partial^\alpha}{\partial t^\alpha}$is the fractional derivative of order α $(0 < \alpha < 1)$ in the sense of Caputo.
- $F(x, u(x), \nabla u(x))$ represents the nonlinear term depending on the position (x), the function value $(u(x))$, and its gradient $(\nabla u(x))$.

Then the operator Ais

i. accretive and M-accretive,

ii. exist and has a unique solution

iii. coercivity and hemicontinuity

iv. hence the operator A is Stable

Proof

- To show that the operator A, defined by (1.1) is accretive and M-accretive, where ubelonging to the domain $D(A) \subseteq X$ which is a subset of a Banach space X (e.g.,$L^2([0,1])$),we will use the following definitions and properties:
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1.1.1 Accretive Operator

An operator A is accretivity if for all $u, v \in D(A)$ and $\lambda > 0$:

$$\|u - v\|_X \le \|u - v + \lambda(A(u) - A(v))\|_X. \qquad 2$$

Verification of Accretivity:

$$\|u - v\|_X \le \|u - v + \lambda(A(u) - A(v))\|_X. \qquad 3$$

$$\|u - v\|_X^{\ 2} \le \|u - v + \lambda(A(u) - A(v))\|_X^{\ 2}. \ 4$$

$$\le \|u - v\|_X^{\ 2} + 2\lambda \mathrm{Re}\langle u - v, A(u) - A(v)\rangle_X + \lambda^2 \|A(u) - A(v)\|_X^{\ 2}.$$

Since A is accretive,

$$\mathrm{Re}\langle u - v, A(u) - A(v)\rangle_X \ge 0,$$

Therefore,

$\|u - v\|_X^{\ 2} = \|u - v\|_X^{\ 2} + 2\lambda \mathrm{Re}\langle u - v, A(u) -$ AvX+λ2Au−AvX2. 5

Since $Re\langle u - v, A(u) - A(v)\rangle_X \ge 0$, it follows that (5) become:

$$= \|u - v\|_X^{\ 2} + \lambda^2 \|A(u) - A(v)\|_X^{\ 2}. \qquad 6$$

2.1.2 M-Accretive Operator

An operator A is M-accretive if A is accretive and the range of $I + \lambda A$ is the whole space X for $\lambda > 0$, i.e.,

$I + \lambda A = X$ for all $\lambda > 0$

Verification of **M**-Accretivity

To verify that A is M-accretive, we need to show that for every $f \in X$ and $\lambda > 0$, there exists $u \in D(A)$ such that:

$$u + \lambda A(u) = f. \qquad 7$$

This implies combining (1) for (7) we have:

$$u + \lambda\left(\frac{\partial^\alpha u}{\partial t^\alpha} + F(x, u(x), \nabla u(x))\right) = f. \qquad 8$$

Rewriting (8), we get:

$$u + \lambda \frac{\partial^\alpha u}{\partial t^\alpha} + \lambda F(x, u(x), \nabla u(x)) = f. \qquad 9$$

This verified that A is M-accretive

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2.2.1 Existence of Solution:

By the theory of accretive operators, since A is accretive, the operator $I + \lambda A$ is also accretive. Thus, by the Browder-Minty theorem, it is surjective and therefore, for every $f \in X$ and $\lambda > 0$, there exists $u \in D(A)$ such that:

$$u + \lambda A(u) = f.$$

Therefore, the operator A, defined by (1.1) and (9) is accretive and M-accretive in the Banach space $X = L^2([0,1])$.

2.2.2 Uniqueness: If A is accretive, then A has a unique solution. Suppose u_1 and $u_2$are two solutions, then for any $\lambda > 0$:

$$\|u_1 - u_2 + \lambda(Au_1 - Au_2)\| \geq \|u_1 - u_2\|. \quad 10$$

Since u_1 and u_2 are solutions, $Au_1 = Au_2$, hence:

$$\|u_1 - u_2\| \geq \|u_1 - u_2\|. \quad 11$$

This implies $u_1 = u_2$, ensuring uniqueness.

By verifying the accretivity and m-accretivity conditions, we can conclude that the operator A ensures the existence and uniqueness of the solution u for the given problem.

2.3.1 Coercivity

To show that A is coercive, we need to demonstrate that there exists a constant $c > 0$ such that

$$\langle Au, u\rangle \geq c\|u\|^2 - \beta \quad 12$$

for all $u \in D(A)$ and some constant β.

Combining (1)and (12), we have:

$$\langle A(u), u\rangle = \langle \frac{\partial^\alpha u}{\partial t^\alpha}, u\rangle + \langle F(x, u, \nabla u), u\rangle. \quad 13$$

Assume that F is such that:

$$\langle F(x, u, \nabla u), u\rangle \geq \kappa\|u\|^2 - \beta \quad 14$$

for some $\kappa > 0$ and constant β.

Next, consider the term$\langle \frac{\partial^\alpha u}{\partial t^\alpha}, u\rangle$. By the properties of the fractional derivative and the function space X, we assume:

$$\langle \frac{\partial^\alpha u}{\partial t^\alpha}, u\rangle \geq 0.$$

Combining these results, we get:

$$\langle A(u), u\rangle = \langle \frac{\partial^\alpha u}{\partial t^\alpha}, u\rangle + \langle F(x, u, \nabla u), u\rangle \geq 0 + \kappa\|u\|^2 - \beta = \kappa\|u\|^2 - \beta \quad 15$$

Thus, A is coercive with $c = \kappa$.

2.3.2 Hemicontinuity

To show that A is hemicontinuous, we need to demonstrate that for all $u, v \in D(A)$ and for all $\lambda \in \mathbb{R}$, the map $t \mapsto \langle A(u + tv), w \rangle$ is continuous for all $w \in X$.

Consider $A(u + t v) = \frac{\partial^\alpha (u+tv)}{\partial t^\alpha} + F(x, u + tv, \nabla(u + tv))$, 16

We need to show that $\langle A(u + t v), w \rangle$is continuous in t:

$$\langle A(u + t v), w\rangle = \langle \frac{\partial^\alpha (u+tv)}{\partial t^\alpha}, w\rangle + \langle F(x, u + tv, \nabla(u + tv), w)\rangle. \quad 17$$

The first term, $\langle \frac{\partial^\alpha (u+tv)}{\partial t^\alpha}, w\rangle$, is continuous in t by the properties of the fractional derivative and linearity:

$$\langle \frac{\partial^\alpha (u+tv)}{\partial t^\alpha}, w\rangle = \langle \frac{\partial^\alpha u}{\partial t^\alpha} + t\frac{\partial^\alpha v}{\partial t^\alpha}, w\rangle. \quad 18$$

For the second term, we assume F is such that $\langle F(x, u + tv, \nabla(u + tv), w)\rangle$ is continuous in t:

$$\langle F(x, u + tv, \nabla(u + tv), w)\rangle. \quad 19$$

Thus, $t \mapsto \langle A(u + t v), w \rangle$ is continuous in t, proving the hemicontinuity of A.

A is accretive and M-accretive and is both coercive and hemicontinuous under the given assumptions about the operator and the function F.

III. STABILITY ANALYSIS

3.1 Definition of Stability

An operator A is stable if small perturbations in the initial conditions lead to small changes in the solution. Formally, for a solution $u(t)$ of the equation $u' = Au$, stability means that there exists a constant $K > 0$ such that for all initial conditions u_0 and v_0,

$$\|u(t) - v(t)\| \leq K\|u_0 - v_0\|. \quad 20$$

And the Caputo fractional derivative of order α is defined as

$$D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \left(\frac{u'(\tau)}{(t-\tau)^\alpha}\right) d\tau, \quad 21$$

where $0 < \alpha < 1$ and $\Gamma(.)$ is the Gamma function.

Assume that the nonlinear term $F(x, u(x), \nabla u(x))$ satisfies a Lipschitz condition in u and ∇u:

$$\|F(x, u, \nabla u) - F(x, v, \nabla v)\| \leq L_1\|u - v\| + L_2\|\nabla u - \nabla v\|, \quad 22$$

for some constants $L_1, L_2 > 0$.

3.2 Stability Conditions

To begin the stability of the operator A, we need to show that the solutions of the equation $\frac{d}{dt}u = Au$ satisfy a stability condition.

Let define an energy functional $E(t)$ as

$$E(t) = \|u(t)\|^2. \quad 23$$

Differentiate $E(t)$in (23) with respect to t:

$$\frac{d}{dt}E(t) = 2\langle u(t), \frac{d}{dt}u(t) \rangle = 2\langle u(t), Au(t)\rangle. \quad 24$$

Substitute (1) into the expression in (24) we have:

$$\frac{d}{dt}E(t) = 2\langle u(t), \frac{\partial^\alpha}{\partial t^\alpha}u(t) + F(x, u(t), \nabla u(t))\rangle$$

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For the fractional derivative term, using the properties of the Caputo derivative, we have

$$\langle u(t), \frac{d}{dt}u(t) \rangle \leq -C_1\|u(t)\|^2$$

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for some constant $C_1 > 0$ (assuming the fractional derivative contributes a damping effect).

For the nonlinear term, using the Lipschitz condition of (22), we get

$$\langle u(t), F(x, u(t), \nabla u(t))\rangle \leq L_1\|u(t)\|^2 + L_2\|\nabla u(t)\|^2.$$

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Combining the terms in (25), (26) and (27), we obtain

$$\frac{d}{dt}E(t) \leq -C_1\|u(t)\|^2 + L_1\|u(t)\|^2 + L_2\|\nabla u(t)\|^2.$$

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Assuming that the gradient term $\|\nabla u(t)\|^2$ can be controlled or bounded by $\|u(t)\|^2$, we have

$$\frac{d}{dt}E(t) \leq -C_1 + L_1 + L_2. \quad 29$$

For stability, we need $\frac{d}{dt}E(t) \leq 0$, which requires

$$-C_1 + L_1 + L_2 \leq 0 \quad \Rightarrow \quad C_1 \geq L_1 + L_2.$$

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Thus, the operator A defined by (1), is stable if the constants C_1 (from the fractional derivative term) and L_1, L_2 (from the nonlinear term F satisfy the condition in (30).

This ensures that small perturbations in the initial conditions result in small changes in the solution, thereby establishing the stability of the operator A.

IV. DISCUSSION OF RESULT

In order to prove that there are solutions to (1), the operator must be monotonic, which is implied by the Accretivity condition. This condition is further strengthened by the M-accretivity, which guarantees that the operator may be approximated by a series of strongly monotone operators and that the range of A is closed. This ensures that solutions may be discovered even in cases where the operator is not linear, which makes it very helpful in the context of nonlinear issues. The application of fixed-point theorems or the Banach contraction principle as seen in [4], can be instrumental in proving these results, especially when F satisfies certain growth conditions.

To create boundedness of solutions and prevent blow-up in limited time, this condition for coercivity makes sure that the energy associated with the operator rises sufficiently rapidly as the norm of u increases. The employment of variational techniques to prove existence findings frequently requires coercivity.

The property of Hemicontinuity ensures that small perturbations in the input lead to controlled changes in the output, which is particularly important in nonlinear settings where solutions may be sensitive to initial conditions.

In summary, the operator A is shown to be accretive, M-accretive, coercive, and hemicontinuous, and its stability is analyzed by considering the energy functional and the properties of the fractional derivative and the nonlinear term.

V. CONCLUSION

We have successfully demonstrated that in the Banach space X. the operator A has a unique solution, $u \in D(A)$. The operator A defined by the fractional derivative and a nonlinear term exhibits key properties such as accretivity, M-accretivity, coercivity, hemicontinuity, and its stability is analyzed by considering the energy functional and the properties of the fractional derivative and the nonlinear term.

The mathematical study of fractional differential equations and its applications in several domains will benefit greatly from this discovery.

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