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Fractional Nonlinear Partial Differential Operators in Closed, Bounded, and Continuous Domain

Echude O[1], Adamaku D[2],Mate O. P[3]
1. Directory of General Studies, Mathematics Unit, Veritas University Abuja, Nigeria,
2. Directory of General Studies, Mathematics Unit, Veritas University Abuja, Nigeria,
3. Directory of General Studies, Mathematics Unit, Veritas University Abuja, Nigeria,
Corresponding Author: EchudeOhiemi

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ABSTRACT: A fractional non-linear partial differential equation with boundary problems was investigated in a closed, bounded and continuous domain by converting such a fractional nonlinear partial differential equation (operator) into an abstract Cauchy problem. The study extends an existing theory in functional analysis to fractional differential, this provided some new insights into non-local interactions and memory effects. The key results were obtained by formulating a mathematical problem for such an operator and using some assumptions, including the solutions' existence, uniqueness, and regularity theorem. Our findings in some way contribute to the growing field of fractional calculus and its applications in various scientific fields.

Keywords: Fractional nonlinear differential equations, Cauchy problem, accretive operators, Banach spaces, existence and uniqueness of solutions

I. INTRODUCTION

In a recent year, fractional calculus has gained significant attention as a powerful mathematical tool because ofits ability to model complex phenomena in various fields, including physics, engineering, biology, and finance [1, 2]. These operators play a crucial role in this study and are characterised by fractional derivativesi.e. there are non-integer order derivatives, which can better provide a more accurate description of complex phenomena compared to the usual traditional integer-order derivatives.In particular, the study of fractional partial differential equations (FPDEs) hasalso gained significant attention among researchers in this field due to their ability to capture non-local and memory effects in system dynamics [3, 4] and this also enriching our understanding of the mathematical framework for describing the behaviour of physical systems that exhibit non-local dynamics and to modal physical processes with accuracy in real life situations.

In this article, FPDEs involving nonlinear operators introduce some layer of complexity and such pose unique challenges in establishing existence and uniqueness solutions. The nonlinearity introduces additional complexities that require specialized techniques and analysis [5, 6].

The fractional nonlinear partial differential operators defined on closed, bounded, and continuous domains are the major point of this research. Our choice of closed, bounded and continuous domains provides a solid framework for the study of fractional non-linear partial differential equations [7]. It is in good alignment with the finite nature of many physical systems and gives the mathematical foundation required for a thorough investigation employing established techniques from functional analysis.Furthermore, the choice of the domain closed, bounded, and continuous, was introduced to help ensure the well-posedness of the fractional nonlinear partial differential equation, this space ensures that mathematical properties hold. The operators in this study will help mathematicians develop tools to analyze solutions of fractional nonlinear partial differential equations.

Our major goal of this article in a closed, bounded, and continuous domain, is to examine the theoretical foundation and practical implications of a fractional non-linear partial differential operator.

This work is an extension of Echudeet al (2017)were‘a non-linear parabolic partial differential equation in a closed, bounded and continuous domain’ is now extended to ‘fractional nonlinear partial differential operators in closed, bounded and continuous domains'.Theyinvestigated on a non-linear parabolic partial differential equation in a closed, bounded and continuous domain which is doneby converting such a non-linear parabolic partial differential equation to an abstract Cauchy problem, the operator was shown to be m-accretive and therefore established that the partial differential equation has a solution by the fundamental results of accretive operators[8].

Recent developments in fractional calculus have also looked into how fractional calculus relates to other mathematical ideas, like measure differential equations and operators of the Stieltjes type [9, 10]. These connections have expanded fractional calculus's field of application and created new directions for the investigation of nonlinear FPDEs.

Fractional Nonlinear Partial Differential Operators in Closed, Bounded, and Continuous Domains is a rapidly developing field of study with a wide range of applications and persistent challenges. This vital field of study is still being advanced by the creation of strong mathematical frameworks and the investigation of innovative problem-solving strategies.

II. PRELIMINARIES

- X is a Banach space equipped with a norm $\|.\|$.
- The operator $A: X \to X$ is a nonlinear operator.
- The function $f: [0, T] \times X \to X$ is a given nonlinear function.
- The initial condition $u(0) = u_0$is given, where $u_0 \in X$.
- Let X is a Banach Space and consider the Caputo fractional derivative $D_t^{\alpha} u(t)$ of order $\boldsymbol{\alpha} \in (\mathbf{0}, \mathbf{1})$is defined by

$$D_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{u'(s)}{(t-s)^{\alpha}} \right) ds.$$

Where $\mathbf{u}'(\mathbf{s})$is the derivative of $\mathbf{u}$ with respect to $\mathbf{s}$
Γ is the Gamma function.

2.1 Mathematical Formulation:

In this article, we focus on analysing on fractional nonlinear partial differential equationsaccretive operator in a closed, bounded, and continuous domain. The mathematical problem can be formulated as follows:

$$\begin{cases} D_t^{\alpha} u(t) = A(u(t)) + f(t, u(t)), & t \in (0, T] \\ u(0) = u_0 & \end{cases} \quad 1$$

D_t^{α} is the Caputo fractional derivative of order $\alpha \in (0,1)$ with respect to time t, A is a nonlinear accretive operator defined on a Banach space X, f is a given nonlinear function.

Theorem 1

Let X be a Banach space, $A: X \to X$ a nonlinear operator, $f: [0, T] \times X \to X$ a nonlinear function, and $\alpha \epsilon (0, 1)$. We consider the Cauchy problem:

$$D_t{}^{\alpha} u(t) = A(u(t)) + f(t, u(t)), \ t \in (0, T] \quad 1.1$$

III. EXISTENCE OF SOLUTIONS TO THEOREM 1

To establish the existence and uniqueness of solutions, we employ the Banach fixed-point theorem. Define the operator $\mathcal{T}: C([0, T] \to C([0, T], X)$ as follows:

Let $C = C([0, T], X)$ be the Banach space of continuous functions from $[0, T]$ to X. Define the operator $\mathcal{T}: C \to C$ as:

$$(\mathcal{T}u)(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(u(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s\alpha-1fs, usds. \quad 2$$

$\mathcal{T}$ is well-defined since A and f are continuous.

To apply the Banach fixed-point theorem (contraction mapping principle), we need to show that the integral operator

$$(\mathcal{T}u)(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A(u(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s\alpha-1fs, usds. \quad 3$$

is a contraction on a suitable complete metric space (typically a Banach space of continuous functions on ([0, T]) with values in (X)).

3.1 Continuity of $\mathcal{T}$:

For $u, v \in C$, we have:

$$\|\mathcal{T}u - \mathcal{T}v\| = \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [A(u(s)) - Avsds+1\Gamma(\alpha)0tt-s\alpha-1[fs, us-fs, vs]ds \quad 4$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|A(u(s)) - A(v(s))\| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, u(s)) - f(s, v(s))\| ds \quad 5$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L \|u(s) - v(s)\| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M \|u(s) - v(s)\| ds \ 6$$

$$\leq \mathbf{L} \|u - v\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \mathbf{M} \|u - v\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \quad 7$$

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$$\leq \mathbf{L}\|\mathrm{u}-\mathrm{v}\| \frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)}+\mathbf{M}\|\mathrm{u}-\mathrm{v}\| \frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)} \quad 8$$

$$\leq (\mathbf{L}+\mathbf{M})\|\mathrm{u}-\mathrm{v}\| \frac{\mathrm{t}^{\alpha}}{\Gamma(\alpha+1)}. \quad 9$$

Therefore, $\mathcal{T}$ is continuous

3.2 Compactness of $\mathcal{T}$:

For $u \in C$, let $B = sup_{t\in[0,T]}\|u(t)\|$. Then:

$\|\mathcal{T}u\| = \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} A\big(u(s)\big)ds + \right.$ $1/\Gamma(\alpha)0tt-s\alpha-1fs, usds$ 10

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|A(u(s))\|ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,u(s))\|ds \quad 11$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\|u(s)\|ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} M\|u(s)\|ds \quad 12$$

$$\leq LB\frac{t^{\alpha}}{\Gamma(\alpha+1)} + MB\frac{t^{\alpha}}{\Gamma(\alpha+1)} \quad 13$$

$$\leq (L+M)B\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \quad 14$$

Therefore, $\mathcal{T}$ maps C to a bounded set in C.

3.4 Fixed Point:

By the Schauder Fixed Point Theorem, $\mathcal{T}$ has a fixed point $u \in C$, which is a solution to the Cauchy problem in (1.1).

3.5 Uniqueness of Solutions to theorem 1

3.5.1 Contradiction:

Suppose there are two solutions $u_1, u_2 \in C$, to the Cauchy problem. Then:

$$D_t{}^{\alpha}(u_1-u_2)(t) = A(u_1(t)) - A(u_2(t)) + f(t,u_1(t)) - f(t,u_2(t)). \quad 15$$

Integrating both sides from 0 to t, we get:

$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[A(u_1(s)) -$ $Au2s]ds+1/\Gamma(\alpha)0tt-s\alpha-1fs,\ \ u1s-fs,u2sds=0.$ 16

By the same argument as in the existence proof, we have:

$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[A(u_1(s)) -$ $Au2s]ds+1/\Gamma(\alpha)0tt-s\alpha-1fs,$ $u1s-fs,u2sds\leq\boldsymbol{L+M}u1-u2t\alpha\Gamma\alpha+1.17$

Since $u_1, u_2 \in C$, we have $\|u_1-u_2\| \leq B$. Therefore (17) become:

$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[A(u_1(s)) -$ $Au2s]ds+1/\Gamma(\alpha)0tt-s\alpha-1fs,$ $u1s-fs,u2sds\leq\boldsymbol{L+MB}t\alpha\Gamma\alpha+1.$ 18

Since $t \in (0,T]$, we have $t^{\alpha} \leq T^{\alpha}$. Therefore (18) become:

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[A(u_1(s)) - A(u_2(s))]ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s,u_1(s)) - f(s,u_2(s))]ds \leq (\boldsymbol{L}+\boldsymbol{M})\boldsymbol{B}\frac{T^{\alpha}}{\Gamma(\alpha+1)}. \quad 19$$

Gronwall's Inequality:

Applying Gronwall's Inequality for fractional differential equations in (19), we obtain:

$$\|u_1(t)-u_2(t)\| \leq 0 \quad \forall t \in [0,T]. \quad 20$$

Hence, $u_1(t) = u_2(t)$.

Under the assumptions that A and f are continuous and satisfy appropriate Lipschitz conditions, the Cauchy problem has a unique solution $u(t)$ in the Banach space X.

Thus, $\mathcal{T}$ is a contraction mapping for sufficiently small T, ensuring the existence of a unique fixed point u, which is the solution to the Cauchy problem.

Thus, we have established the existence and uniqueness of the solution $u \in C([0,T],X)$ to the Cauchy problem (1.1)

IV. REGULARITY OF SOLUTIONS TO THEOREM 1

Under the above assumptions, we can establish the existence and regularity of solutions to the fractional nonlinear PDE with an accretive operator.

To prove the Regularity Theorem for the solution u of the Cauchy problem (1)

$$D_t{}^{\alpha}u(t) = A(u(t)) + f(t,u(t)),$$
$$t \in (0,T] \quad u(0) = u_0$$

where $D_t{}^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $A: X \to X$ is a nonlinear operator on the Banach space X, and $f: [0,T] \times X \to X$is a nonlinear function, we aim to show that $u \in C([0,T];\ X) \cap C^1((0,T];X)$,we assume the following regularity conditions:

Assumption 1 (Initial Condition Regularity):

$$u_0 \in D(A),$$

where $D(A)$ denotes the domain of the operator A.

Assumption 2 (Regularity of A):

$A: D(A) \subset X \to X$ is a nonlinear operator satisfying:

1. **Local Lipschitz Continuity:**

$$\| A(u) - A(v) \| \leq L_A \| u - v \|, \forall u, v \in D(A),$$

Where L_A is a Lipschitz constant.

2. **Growth Condition:**

$$\| A(u) \| \leq K_A(1 + \| u \|), \forall u \in D(A),$$

Where$K_A$is a positive constant.

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Assumption 3 (Regularity of f):
$f: [0, T] \times X \to X$ is a continuous function satisfying:

1. **Local Lipschitz Continuity in u:**
$\| f(t,u) - f(t,v) \| \leq L_f(t) \| u - v \|,$
$\forall t \in [0,T], \forall u, v \in X,$
where $L_f(t)$is a Lipschitz constant that may depend on t.

2. **Growth Condition:** $\| f(t,u) \| \leq K_f(1 + \| u \|),$
$\forall t \in [0,T]$, $\forall u \in X$,
where K_f is a positive constant.

Assumption 4 (Regularity of the Caputo Fractional Derivative):
The Caputo fractional derivative $D_t^{\alpha} u(t)$ of order $\alpha \in (0,1)$is defined by

$$D_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{u'(s)}{(t-s)^{\alpha}} \right) ds.$$

Where $u'(s)$ denotes the derivative of u with respect to s and Γ is the Gamma function.
For $u \in C^1([0,T], X)$ (the space of continuously differentiable functions from $[0,T]$ to X), the Caputo fractional derivative $D_t^{\alpha} u(t)$exists and is continuous on $(0,T]$.

Proof:
4.1 Continuity on $[0,T]$
Since u is a solution to the differential equation (1), it follows that $u \in C([0,T]; X)$. This is because $D_t^{\alpha} u(t)$, $A(u(t))$, and $f(t, u(t))$ are all continuous functionsof t on $[0,T]$by equation (4 – 9) and hence $u(t)$ inherits this continuity of equation (4 – 9).
4.2 Differentiability on $[0,T]$
To establish $u \in C^1((0,T]; X)$, we need to show that $u'(t)$ exists and is continuous on $(0,T]$.
Existence of $u'(t)$:
The Caputo fractional derivative $D_t{}^{\alpha} u(t)$ is defined by

$$D_t{}^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{u'(t)}{(t-s)^{\alpha}} \right) ds + \frac{u(0)}{\Gamma(1-\alpha)} t^{(1-t)}.$$

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For $u \in C([0,T]; X)$, the right-hand side is well-defined and continuous on $(0,T]$. Therefore, $u'(t)$ exists for $t \in (0,T]$.
4.3 Continuity of $u'(t)$:
Since u is continuous on $[0,T]$, $u'(t)$ is the pointwise limit of a sequence of continuous functions (constructed from the definition of $D_t{}^{\alpha} u(t)$), implying $u'(t)$ itself is continuous on $(0,T]$.
Therefore, u satisfies $u \in C([0,T]; X)$ and $u \in C^1((0,T]; X)$. This completes the proof of the Regularity Theorem.

By demonstrating the continuity of u on $[0,T]$ and the existence and continuity of $u'(t)$ on $(0,T]$, we have established that $u \in C([0,T]; X) \cap C^1((0,T]; X)$. This regularity ensures that u is sufficiently smooth for further mathematical analysis and applications in the context of fractional differential equations.

The regularity of solutions depends on the properties of the operator A and the function f. If A and f are sufficiently smooth, the solution $u(t)$ inherits this regularity. More specifically, if A and f are both continuous and differentiable, then $u(t)$ is continuously differentiable in t.Stronger smoothness assumptions on A and f can produce higher-order regularity.

V. DISCUSSION OF RESULTS

Using the Banach fixed point theorem, we proved the existence and uniqueness of a solution $u \in C([0,T], X)$ to the Cauchy problem (1.1) for the results to hold

Considering a significant and ongoing topic of research in fractional calculus, this finding offers a strong theoretical framework for analysing the Cauchy issue for fractional nonlinear partial differential equations in Banach spaces [1]. These findings offer a thorough foundation for comprehending how fractional Cauchy problem solutions behave in Banach spaces. Rich mathematical structures and behaviours result from the interaction of the fractional derivative, the nonlinear operator A, and the nonlinear function f.

VI. EXAMPLES AND APPLICATIONS

We illustrate the following examples of differential equations involving the Caputo fractional derivative D_t^{α} of order $\alpha \in (0,1)$, applied to a Banach space X, and their applications to demonstrate the applicability of the resulting conclusions.

- Consider the fractional diffusion equation:

$$D_t{}^{\alpha} u(t) = \Delta u(t) + f(u(t)),$$
$$t \in (0,T] \quad u(0) = u_0,$$

where Δ denotes the Laplace operator on X, and $f: X \to X$ is a nonlinear function.
This is an example of the equation model's anomalous diffusion phenomena where $D_t^{\alpha} u(t)$ represents the fractional derivative of $u(t)$ with respect to time t.

- Consider the fractional reaction-diffusion equation:

$$D_t{}^{\alpha} u(t) = \Delta u(t) + g(u(t)),$$
$$t \in (0,T] \quad u(0) = u_0,$$

where Δ is the Laplace operator on X, and $g: X \to X$ is a nonlinear reaction term [11]

This equation is used in mathematical biology to model population dynamics where the population density $u(t)$ diffuses through space and undergoes nonlinear reactions governed by $g(u(t))$.

- Consider the fractional Allen-Cahn equation:

$$D_t^{\alpha} u(t) = -\Delta u(t) + u(t) - u(t)^3,$$
$$t \in (0,T] \quad u(0) = u_0.$$

where Δ denotes the Laplace operator on X. [12]
Phase transitions and pattern development in materials are described by this equation in the field of materials science. The Laplacian of $u(t)$ is represented by the expression $-\Delta u(t))$, and nonlinear interactions are introduced by $u(t) - u(t)^3$.Think about the Burgers' equation for fractions:

- Think about the Burgers' equation for fractions:

$$D_t^{\alpha} u(t) + u(t) D_x u(t) = A(u(t)),$$
$$t \in (0,T] \quad u(0) = u_0,$$

where $A \downarrow X \to X$ is a nonlinear operator and D_x is the spatial derivative.
In fluid dynamics, this equation is used to simulate the behaviour of nonlinear shocks and waves. The convective term is represented by the term $u(t)D_x u(t)$, while nonlinear interactions are introduced by the term A(u(t)).

In conclusion, these examples show how Caputo fractional derivatives can be used to simulate a variety of processes in a range of scientific fields, including as materials science, physics, and biology. The solutions $u(t)$ dwell in a flexible framework made possible by the choice of Banach space X, which guarantees that the solutions are well-defined and appropriately regular for analysis in these complicated systems.

VII. CONCLUSION

The Cauchy problem (1), which involves the Caputo fractional derivative D_t^{α}, a nonlinear operator A on X, and a nonlinear function f, has a unique solution, $u \in C([0,\mathrm{T}]; \mathrm{X})$, according to the reasoning presented above. Moreover, it is shown that the solution u is sufficiently regular, providing that it continues and proper differentiability with regard to t.

The results of this research provide a strong basis for applications in numerous scientific and engineering domains by facilitating the analysis and solution of fractional differential equations in Banach spaces. We lay a strong foundation for theoretical study and real-world applications in science and engineering by proving the existence, uniqueness, and regularity of solutions.

Fractional differential equations (FDEs), such as the one in equation (1), are being employed more and more to explain complicated processes that occur in a variety of domains and applications. More accurate and flexible portrayals of memory and heredity qualities of materials and processes are possible because to the fractional order α [1]. The ideas of Infinite-dimensional issues are possible in the Banach space setting, which is important for many physical models [13].

Equations like the one in (1) can be used in control theory to characterise systems having memory effects, which can result in novel control schemes and optimisation issues [14].

Numerous difficult facets of contemporary analysis and its applications are captured in this problem. Research on it advances theoretical mathematics as well as useful models in a number of scientific fields.

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