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Quadratic and Cubic Spline Functions and Some Applications

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ABSTRACT
In general, the proposed quadratic spline interpolation theory yields superior fits to continuous functions than the current cubic spline interpolation technique. In this study, quadratic and cubic spline regressions are explained and applied on 2 numerical examples. Graphs of observation values are drawn. The parameter coefficients of quadratic and cubic spline regressions were obtained using MATLAB programming. Quadratic and cubic spline regressions are useful methods that can be applied to many observation data.

Keywords: Quadratic spline, cubic spline, observation, graph.

I. INTRODUCTION

Mathematicians utilize spline functions as a method for fitting curves (Ahlberg et al., 1967). They were recommended for piecewise regression by Poirier (1973). He demonstrated how to use LS-estimators of the joint point ordinates to estimate the parameters of Cubic Splines. Sadly, one is presented with a linear equation system that is not complete rank, meaning that in order to obtain a unique solution, one must identify tenable linear independent conditions. This approach is based on the widespread application of splines in physics, where fitting a smooth curve through predetermined fixed points is the challenge.

Based on how they are applied in mathematics and physics when fitting smooth curves via specified fixed points is the goal. A technique for estimating such points is proposed by Poirer (1973). In a study, it is shown that it is possible to estimate the parameters of a Spline directly from the data by the Least Square Estimator. Cubic, quadratic and linear Splines were used as regression functions (Gnad, 1977).Mulla (2007) displayed the spline function summarizing the association between serum albumin and the crude risk of hospital mortality.

Piecewise polynomials of degree n are the definition of spline functions. The fragments come together to form what are known as knots and satisfy continuity requirements for both the function and the first n-1 derivatives. Consequently, a continuous function with n-1 continuous derivatives is a spline function of degree n (Wold, 1974).

Since spline models take into account variation in the outcome's risk within the categories of interest, Witte and Greenland (1997) claim that they are superior to most categorical studies in the evaluation of dose-response. A decent approximation to more complex techniques like penalized splines and nonparametric regression may be obtained using splines.

In a work, Kohli (1978) demonstrated how a profit function technique might be used to estimate the input (and import) demand functions and the output (and export) supply functions for a market economy econometrically. Using data from Canada on six different goods consumption, investment, exports, imports, labor, and capital he computed the parameters of a translog variable profit function. The right curvature criteria on the variable profit function, however, were no longer met when he raised the quantity of commodities. Therefore, in a different work, Kohli (1989) estimated flexible functional forms for variable profit functions that had the advantage of allowing the required curvature constraints to be globally enforced without compromising the functional form's flexibility. His functional forms for the variable profit function were variations of the normalized quadratic functional form, and he used data from the United States on the six listed items.

The aim of this study is to apply quadratic and cubic spline functions on several sample data and evaluate the results.

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II. METHODS

Quadratic spline regression

Presence and uniqueness of quadratic spline

Given that s(x) is a polynomial with a maximum degree of 2 in every subinterval $[x_i, x_{i+1}]$,

$$s(x) = P_i(x) = a_i(x - x_i)^2 + b_i(x - x_i) + c_i$$
$$x_i \epsilon [x_i, x_{i+1}], \quad i = 0(1)n - 1$$

The following consistency relation is widely known for uniform division of the interval for a quadratic spline (Loscalzo, 1969; Loscalzo and Talbot, 1967).

$$s'_{i+1} + s_i = \frac{2}{h}(s_{i+1} - s_i), \quad 0 \le i \le n - 1$$

Then, the coefficients of s(x) are determined from the relations

$$a_i = \frac{(s_{i+1} - s_i - hs'_i)}{h^2}, \quad b_i = s'_i, \quad c_i = s_{i,} \quad 0 \le i \le n - 1.$$

Through the evaluation or differentiation of the associated quadratic spline polynomial, one may determine the approximate values of y(x) and its derivative at the sites that are not knots.

The local truncation error is given by

$$\begin{cases} y'_{i+1} + y'_i = \frac{2}{h}(y_{i+1} - y_i) + \frac{h^2}{6} y'''(\alpha_i), \\ x_i < \alpha_i < x_{i+1}, \quad 0 \le i \le n - 1 \end{cases}$$

It is not possible to enhance the assertion $\|e\|=(h^3)$ for quadratic spline interpolation. Nonetheless, point error limits with an additional factor of h are valid on a uniform mesh with $x_i=i/n$ and $h_i=h=1/n$. These bear similarities to T.R. Lucas's (1972) point error boundaries for cubic spline interpolation (Birkhoff and de Boor, 1965).

Cubic spline regression

Cubic spline regression: "k + 3" is the number of parameters necessary, with the exception of the a parameter, and there is no endpoint requirement (Stone and Koo, 1985). Regression of the one (α) knots cubic spline function in this instance happens as follows:

$$Y(t) = a + bx + cx^2 + dx^3 + e\,(x - \alpha)^3$$

Model of cubic spline regression:

$$y = a + bx + cx^2 + dx^3 + \sum_{i=1}^{k} k(x - x_i)^3$$

Where, x refers to independent variable; a, b, c, c and d are parameter coefficients to be estimated, and k is the number of knots in the splines.

III. RESULTS

The application that obtains quadratic and cubic spline regressions by giving the observation values of two functions is summarized as follows.

Example 1: Let the observation values of the x and y variables forming the $y = x^2 + 5x$ function be generated. The y values for x = -2, -1, 0, 1, 2 and 3 are given in Table 1.

Table 1. X and y observation values

x	-2	-1	0	1	2	3
y	-6	-4	0	6	14	24

The graph of x and y variables given in Table 1 was created and presented in Figure 1.

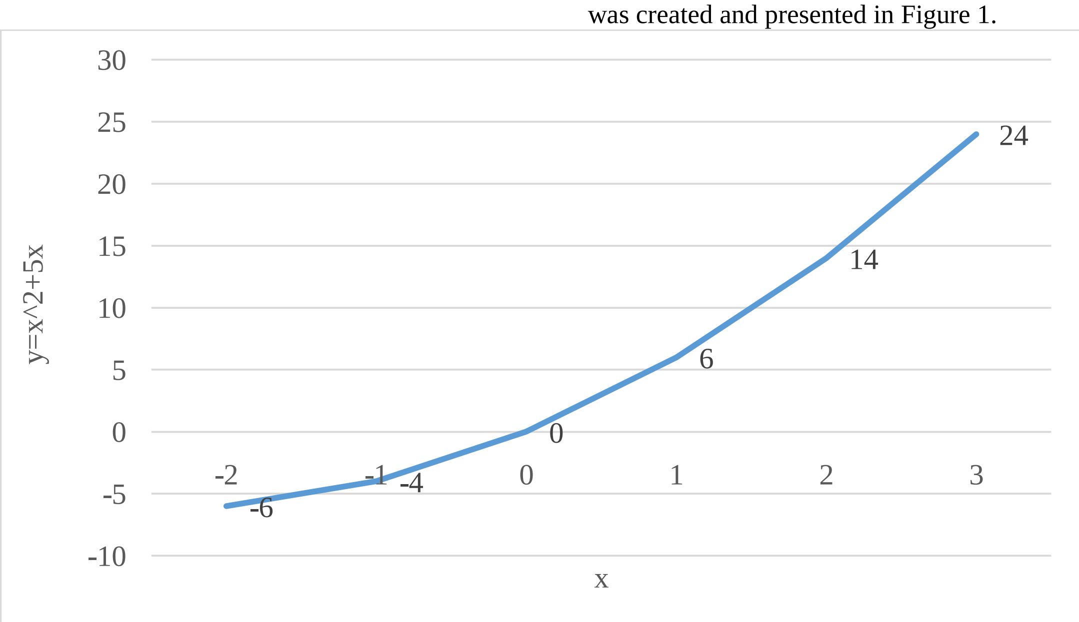

Figure 1. Graph of the function $y = x^2 + 5x$

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When it is desired to create quadratic spline regression from the observation values of the function $y = x^2 + 5x$, the codes in Figure 2 were written in MATLAB programming.

```
closeall;
T=[-2 -1 0 1 2 3]';
beta=[-6 -4 0 6 14 24]';
alpha=10^1*[-.52 -.43  .03  .51 1.3 2.7]';
Ti=[-1.6 -0.6 0.75 1.5 2.6]';
PropertyC=interp1(T,[beta,alpha],Ti,'spline');
[Ti  PropertyC]
plot(T,alpha,'*-',T,beta,'*-',Ti,PropertyC(:,1),'o',Ti,PropertyC(:,2),'o')
legend('\alpha','\beta','New  \beta','New \alpha',2);
xlabel('X,  T');
ylabel('Y  \beta  anddiffusivity  \alpha');
```

Figure 2. MATLAB program commands for quadratic spline regression

The programming output is as follows. Untitled
ans =

-1.6000	-5.4400	-5.6168
-0.6000	-2.6400	-2.6128
0.7500	4.3125	3.7672
1.5000	9.7500	8.4500
2.6000	19.7600	20.5016

The quadratic spline regressions obtained using these coefficients are as follows.

$$y = -1.6\ (x+2)^2 - 5.4\ (x+2) - 5.6168 \qquad -2 < x < -1$$

$$y = -0.6\ (x+1)^2 - 2.64\ (x+1) - 2.6128 \qquad -1 < x < 0$$

$$y = 0.75\ x^2 + 4.3125\ x + 3.7672 \qquad 0 < x < 1$$

$$y = 1.5\ (x-1)^2 + 9.75\ (x-1) + 8.45 \qquad 1 < x < 2$$

$$y = 2.6\ (x-2)^2 + 19.76\ (x-2) + 20.5016 \qquad 2 < x < 3$$

The graph obtained as a result of the codes in MATLAB programming is shown in Figure 3.

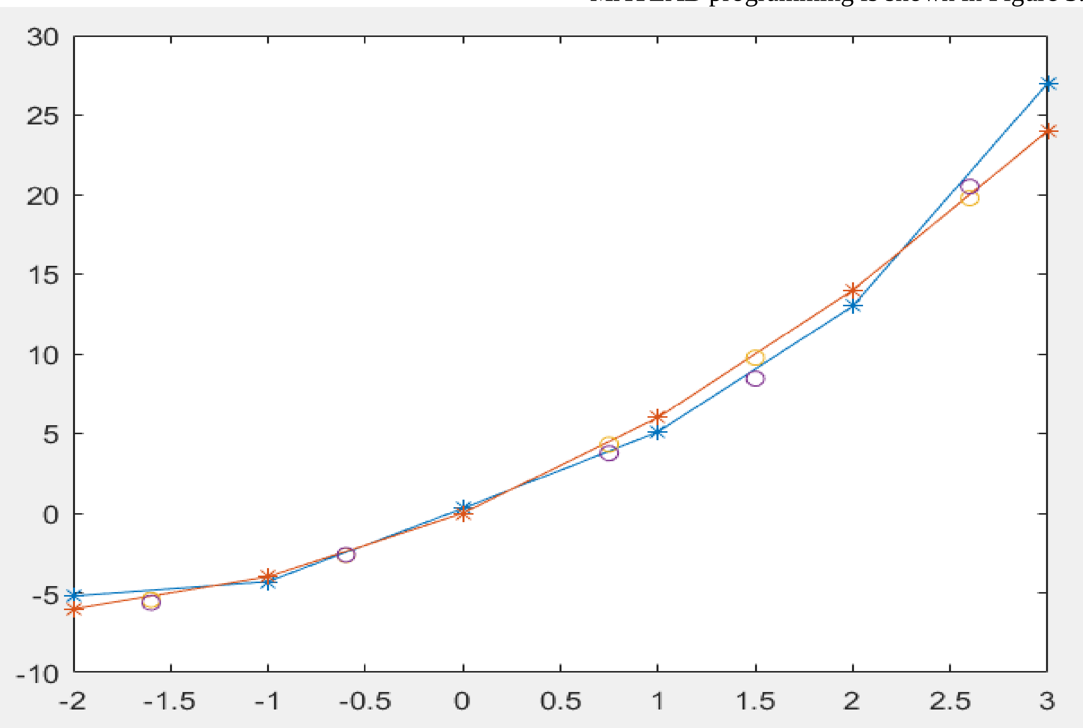

Figure 3. Observation evaluated quadratic spline graph

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MATLAB codes were written to generate cubic spline regression for the same observation values presented in Table 1 (Figure 4).

```
closeall;
x=[-2,-1,0,1,2,3];
y=[-6,-4,0,6,14,24];
plot(x,y,'.','markersize',20);
xx=linspace(0,6);
pp=spline(x,[-6,y,24]);
yy=ppval(pp,xx);
holdon;
plot(xx,yy);
pp.coefs
pp.coefs
spline(x,y,4)
```

Figure 4. MATLAB programming commands for cubic spline regression

The programming output of Figure 4 is as follows.

```
ans =
  -5.0622  13.0622  -6.0000  -6.0000
   1.1866  -2.1244   4.9378  -4.0000
   0.3158   1.4354   4.2488        0
  -2.4498   2.3828   8.0670   6.0000
   9.4833  -4.9665   5.4833  14.0000
ans =   36.0000
```

In the 'spline(x,y,4)' command, y=36 for x=4. The following cubic spline functions are obtained from the above coefficients.

$$y = -5.0622(x+2)^3 + 13.0622\,(x+2)^2 - 6\,(x+2) - 6 \qquad -2 < x < -1$$

$$y = 1.1866(x+1)^3 - 2.1244(x+1)^2 + 4.9378(x+1) - 4 \qquad -1 < x < 0$$

$$y = 0.3158x^3 + 1.4354x^2 + \ 4.2488\,x \qquad 0 < x < 1$$

$$y = -2.4498\,(x-1)^3 + 2.3828(x-1)^2 + 8.067(x-1) + 6 \qquad 1 < x < 2$$

$$y = 9.4833\,(x-2)^3 - 4.9665(x-2)^2 + 5.4833(x-2) + 14 \qquad 2 < x < 3$$

The cubic spline regression graph obtained as a result of the programming in Figure 4 is presented in Figure 5.

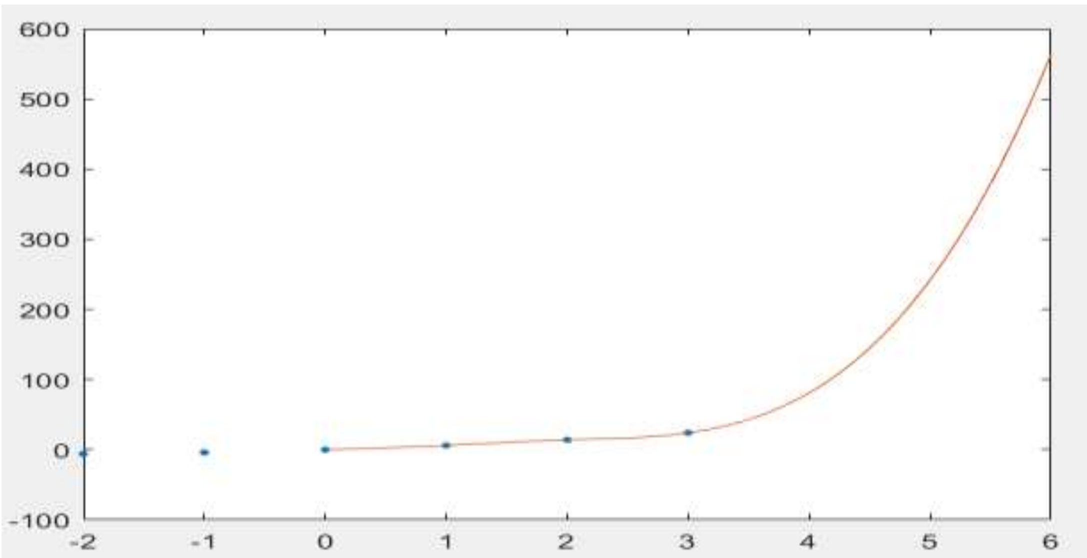

Figure 5. Cubic spline function (for $y = x^2 + 5x$)

Example 2: The observation values of the function $y = 2x^3 - 3$ are given in Table 2 and the related graph is given in Figure 6.MATLAB programming codes for generating quadratic spline function are presented in Figure 7.

Table 2. Data for the function $y = 2x^3 - 3$

x	-2	-1	0	1	2	3	4
y	-19	-5	-3	-1	13	51	125

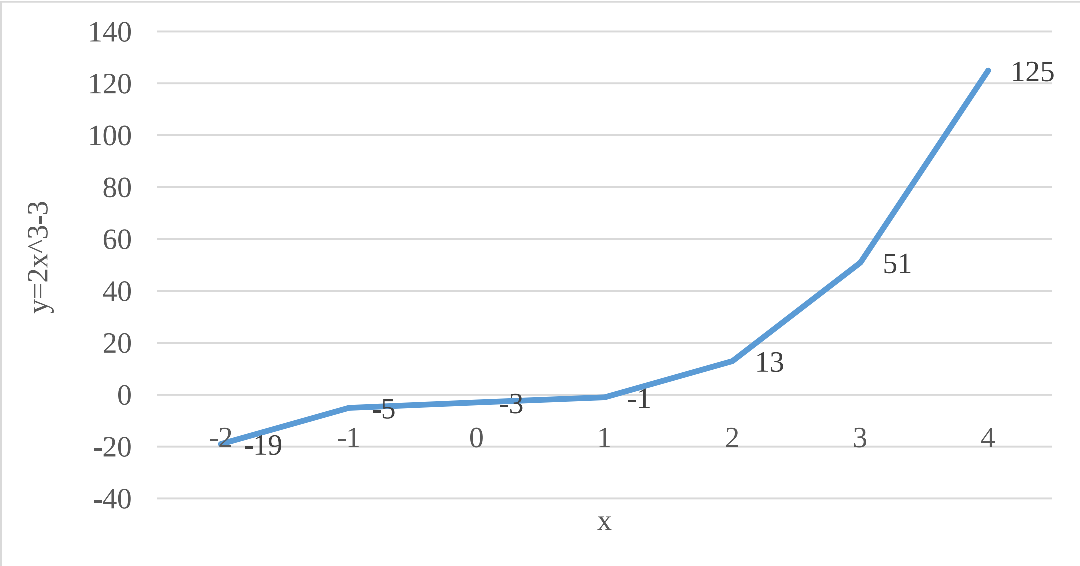

Figure 6. $y = 2x^3 - 3$function graph

```
closeall;
T=[-2 -1 0 1 2 3 4]';
beta=[-19 -5 -3 -1 13 51 125]';
alpha=10^1*[-.16 -.4  -.2  -.1 1.1 4.6 11.9]';
Ti=[-1.6 -0.6 0.75 1.5 2.6 3.5]';
PropertyC=interp1(T,[beta,alpha],Ti,'spline');
[Ti  PropertyC]
plot(T,alpha,'*-',T,beta,'*-',Ti,PropertyC(:,1),'o',Ti,PropertyC(:,2),'o')
legend('\alpha','\beta','New  \beta','New \alpha',2);
xlabel('X,  T');
ylabel('Y\beta  anddiffusivity  \alpha');
```

Figure 7. MATLAB program commands that create quadratic spline regression.

The MATLAB program output for the $y = 2x^3 - 3$function is as follows.

ans =

```
  -1.6000  -11.1920   -3.7353
  -0.6000   -3.4320   -3.1616
   0.7500   -2.1562   -1.7080
   1.5000    3.7500    2.8536
   2.6000   32.1520   28.3424
   3.5000   82.7500   76.7429
```

These coefficients can be written as a quadratic spline function as follows.

$$y = -1.6\,(x+2)^2 - 11.192\,(x+2) - 3.7353 \qquad -2 < x < -1$$

$$y = -0.6\,(x+1)^2 - 3.43\,(x+1) - 3.1616 \qquad -1 < x < 0$$

$$y = 0.75\,x^2 - 2.1562\,x - 1.708 \qquad 0 < x < 1$$

$$y = 1.5\,(x-1)^2 + 3.75\,(x-1) + 2.8536 \qquad 1 < x < 2$$

$$y = 2.6\,(x-2)^2 + 32.152\,(x-2) + 28.3424 \qquad 2 < x < 3$$

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$$y = 3.5\ (x-3)^2 + 82.75\ (x-3) + 76.7429 \qquad 3 < x < 4$$

MATLAB codes are presented to create a cubic spline function from the function data given in Table 2 (Figure 8).

```
closeall;
x=[-2,-1,0,1,2,3,4];
y=[-19,-5,-3,-1,13,51,125];
plot(x,y,'.','markersize',10);
xx=linspace(-2,4);
pp=spline(x,[-19,y,125]);
yy=ppval(pp,xx);
holdon;
plot(xx,yy);
pp.coefs
spline(x,y,5)
```

Figure 8. MATLAB commands of the cubic spline regression generated from the $y = 2x^3 - 3$function

The output of this program is as follows.

```
ans =
  -29.5154   62.5154  -19.0000  -19.0000
   10.5462  -26.0308   17.4846   -5.0000
   -0.6692    5.6077   -2.9385   -3.0000
    4.1308    3.6000    6.2692   -1.0000
   -3.8538   15.9923   25.8615   13.0000
   23.2846    4.4308   46.2846   51.0000
ans =   247
```

In the “spline(x,y,5)” command, y=247 was found for x=5. The following cubic spline functions are obtained from the above coefficients.

$$y = -29.5154(x+2)^3 + 62.5154(x+2)^2 - 19\ (x+2) - 19 \qquad -2 < x < -1$$

$$y = 10.5462\ (x+1)^3 - 26.0308(x+1)^2 + 17.4846(x+1) - 5 \qquad -1 < x < 0$$

$$y = -0.6692x^3 + 5.6077x^2 - 2.9385x - 3 \qquad 0 < x < 1$$

$$y = 4.1308\ (x-1)^3 + 3.6(x-1)^2 + 6.2692(x-1) - 1 \qquad 1 < x < 2$$

$$y = -3.8538(x-2)^3 + 15.9923(x-2)^2 + 25.8615(x-2) + 13 \qquad 2 < x < 3$$

$$y = 23.2846(x-3)^3 + 4.4308(x-3)^2 + 46.2846\ (x-3) + 51 \qquad 3 < x < 4$$

The cubic spline regression graph created with the MATLAB program is presented in Figure 9.

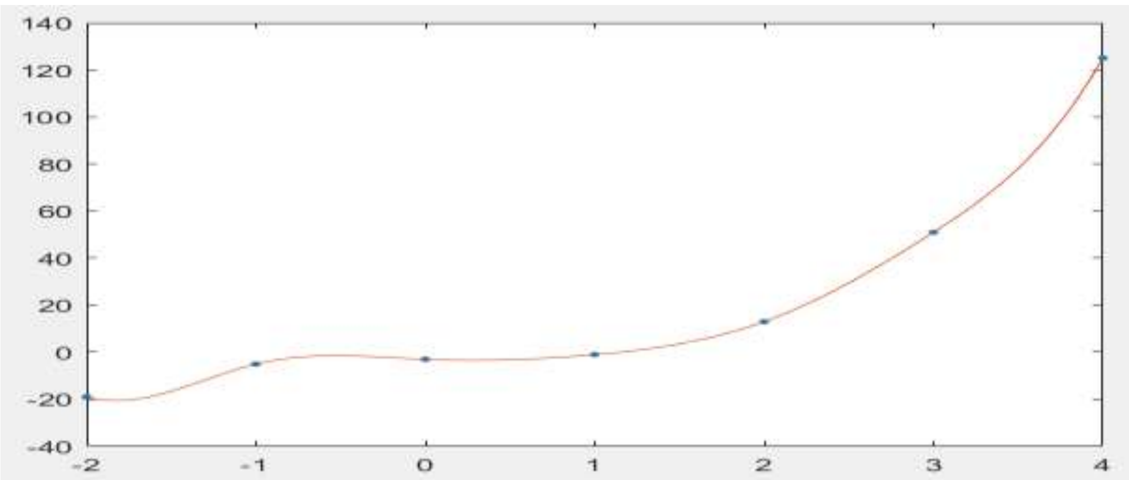

Figure 9. Cubic spline function (for $y = 2x^3 - 3$)

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IV. DISCUSSION

In a study, [0, l] was partitioned into subintervals $h_1, h_2, \dots, h_n$. P_n was an associated cubic spline interpolation operator defined on the space C[0,1].

$$h_0 = h_n \text{and} \quad m_n = \max\left\{\frac{h_i}{h_j} : |i - j| = 1\right\}.$$

Examples were given for which m,, is uniformly bounded as n tends to infinity while $\|P_n\|$was unbounded. The periodic cubic spline interpolation operator was shown to have uniformly bounded norm if$m_n \leq 2.439$ for all n (Marsden, 1974).

In a different study, cubic spline interpolation was used in a revolutionary categorization technique. Regardless of how random the data appeared, splines linked it successfully and efficiently. Interpolating data became simple when the spline generating technique was developed. Additionally, it was observed that statistical classification techniques need less time than created neural network techniques since statistical techniques don't require learning, but designed neural networks must voice emotion detection systems may make use of the comparable cubic spline interpolation behavior of a group of voice signals that represent different emotions (Abdulmohsin et al., 2022).

In the study of Karim et al. (2014),cubic spline interpolation was performed for petroleum engineering data.It was applied cubic spline interpolation on [2,3], [3,4] and [4,5] (time (second)) in car accelerating data.It was discussed the error analysis for cubic spline interpolation and Piecewise Cubic Hermite Spline (PCHIP). It was used three data sets taken from true function$f(x) = x^2 - 8$, $f(x) = 2e^x - x^2$ and $f(x) = \cos 10(x)$respectively. For the error measurements it was used Absolute error and Root Mean Square Error (RMSE). RMSE values for both interpolation methods were calculated as very small. This is a statistically good result.

V. CONCLUSION

This paper discussed the use of quadratic spline interpolation and cubic spline interpolation for $y = x^2 + 5x$ and $y = 2x^3 - 3$functions.It was possible to infer from the numerical findings that the final interpolating curves' form and the observation data were identical. Compared to interpolating curves created by employing quadratic spline interpolation, cubic spline interpolation produces more smooth interpolating curves. Overall both quadratic spline and cubic spline worked well for all tested data sets of functions.

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