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Strong Convergence Theorems With Three Steps Iteration Process For Generalized Asymptotically Quasi-Nonexpansive Mappings In CAT (0) Spaces

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ABSTRACT
In this paper, we establish strong convergence theorems with three steps iteration process for generalized asymptotically quasi-nonexpansive mappings in CAT(0) space. Our results extend, generalize and improve many well-known results in the literature.
Keywords: CAT(0) space, Generalized asymptotically quasi-nonexpansive mapping; strong convergence.

I. INTRODUCTION

Consider (X, d) as a metric space. A geodesic path between $x \in X$ and $y \in X$ (or, to put it another way, a geodesic from x to y) is a map c from $[0, l]$ to X with $c(0) = x$, $c(l) = y$ and $(d(c(t), c(t_0)) = |t-t_0|$, for any $t, t_0 \in [0, l]$. Thus c is an isometry and $d(x, y) = l$. The image of c is a geodesic (or metric) segment that joins x and y. When geodesic is unique, it is denoted by [x, y].

The space (X, d) is said to be a geodesic space if every two points in X are connected by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic between x and y for any x, y in X. If D contains every geodesic segment connecting any two points, the subset $D \subseteq X$ is convex.

A geodesic triangle $4(x_1, x_2, x_3)$ in a geodesic metric space (X, d) is made up of three points x_1, x_2, x_3 in X (the vertices of 4), with a geodesic segment connecting each pair of vertices (the edge of 4). A compari son triangle for the geodesic triangle $4(x_1, x_2, x_3)$ in (X, d) is a triangle $_4(x_1, x_2, x_3) = 4(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in Euclidean space R^2such that $d_R2(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for i, j ∈ {1, 2, 3} [1].

If the distance between any two points on a geodesic triangle *4* does not exceed the distance between its corresponding pair of points on its compar ison triangle $\bar{4}$, then the geodesic space X is a $CAT(0)$ space.

Let $\bar{4}$ be a comparison triangle for a geodesic triangle 4 in X. The 4 is satisfy the CAT(0) inequality if ∀ x, y ∈ 4 and all comparison points $\bar{x}, \bar{y} \in \bar{4}$ such that

$$d(x, y) \leq d_R2(\bar{x}, \bar{y}). \quad (1.1)$$

A complete CAT(0) space is often called Hadamard space [9] . Let x, y, z are points of X and y_0 be the midpoint of segment [y, z], denoted by $\frac{y \oplus z}{2}$, then the $CAT(0)$ inequality gives

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z). \quad (1.2)$$

This is called (CN) inequality of Bruhat and Tits [2]. A geodesic space is said to be $CAT(0)$ space if and only if it satisfies the (CN) inequality [1].

Fixed point theory in $CAT(0)$ spaces was first studied by Kirk [6]. He proved that every nonexpansive mapping defined on closed, bounded convex subset of a complete $CAT(0)$ space always had a fixed point.

Let D be a non-empty subset of a $CAT(0)$ space X and let $T : D \to D$ be a mapping, In 2016, Sintunavarat and Pitea[12] introduced an iteration scheme in connection with Berinde-type operators. For an arbitrary $x_1 \in D$, a se quence $\{x_n\}$ results an output of the following three-step procedure:

$$\begin{cases} y_n = (1-b_n)x_n + b_n T x_n \\ z_n = (1-c_n)x_n + c_n y_n \\ x_{n+1} = (1-a_n)T z_n + a_n T y_n, \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in (0, 1).
For CAT(0) spaces, the iteration procedure S_n can be adapted as follows: for an arbitrary $x_1 \in D$, let $\{x_n\}$ be generated by the procedure below:

$$\begin{cases} y_n = (1-b_n)x_n \oplus b_n T x_n \\ z_n = (1-c_n)x_n \oplus c_n y_n \\ x_{n+1} = (1-a_n)T z_n \oplus a_n T y_n, \ n \in \mathbb{N}. \end{cases} \tag{1.4}$$

This iteration process can be presented in connection with two Garcia-Falset operators, as follows: for an arbitrary $x_1 \in D$, let $\{x_n\}$ be generated by the procedure:

$$\begin{cases} y_n = (1-b_n)x_n \oplus b_n T^n x_n \\ z_n = (1-c_n)x_n \oplus c_n y_n \\ x_{n+1} = (1-a_n)T^n z_n \oplus a_n T^n y_n, \ n \in \mathbb{N}. \end{cases} \tag{1.5}$$

II. PRELIMINARIES

Let us recall some definitions and known results in the existing literature on this concept.
Definition 2.1. Let (X, d) be a metric space and D its nonempty subset. Let $T : D \to D$ be a mapping. A point $x \in D$ is called a fixed point of T if $T x = x$. We will also denote by $S_f(T)$ the set of fixed points of T , that is , $S_f(T) = \{x \in D : T x = x\}$.
Definition 2.2. Let (X, d) be a CAT(0) space and D be its nonempty subset of X in CAT(0) space.
Then $T : D \to D$ is said to be

1. nonexpansive, if $d(T x, T y) \le d(x, y)$, for all $x, y \in D$;
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2. uniformly L-Lipschitzian, if there exists a $L \in (0, \infty)$ such that $d(T^n x, T^n y) \le Ld(x, y)$ for all $x, y \in D$ and $n \ge 1$;
3. generalized asymptotically quasi-nonexpansive [8] if $S_f(T)$ 6= ∅ and there exists two sequences $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \kappa_n$ such that $d(T^n x, p) \le (1 + \mu_n)d(x, p) + \kappa_n$, for all $x \in D$, $n \in N$;
4. compact, if for any bounded sequence $\{x_n\}$ in D, the sequence $\{T x_n\}$ contains a converging subsequence.

Lemma 2.1. [11] Let X be a CAT(0) space .
(i) Let $x, y \in X$, For each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \ d(y, z) = (1 - t)d(x, y). \tag{2.1}$$

We use the notation $(1 - t)x \oplus ty$ for unique point z satisfying (2.1)

(ii) For all $t \in [0, 1]$ and $x, y, z \in X$

$$d(((1-t)x \oplus)ty, z) \le (1-t)d(x, z) + td(y, z). \tag{2.2}$$

(iii) For all $t \in [0, 1]$ and $x, y, z \in X$

$$d^2(((1-t)x \oplus)ty, z) \le (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y). \tag{2.3}$$

Let $\{x_n\}$ be a sequence in a metric space (X, d), and D be a subset of X. We say the $\{x_n\}$ is of monotone type (A) with respect to D if each $p \in D$, there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of non negative real numbers such that $\sum_{n=1}^{n=\infty} \alpha_n < \infty$ and $\sum_{n=1}^{n=\infty} \beta_n < \infty$ such that

$$d(x_{n+1}, p) \le (1 + \alpha_n)d(x_n, p) + \beta_n, \tag{2.4}$$

we say the $\{x_n\}$ is of monotone type (B) with respect to D [13] if each $p \in D$, there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of non negative real numbers

such that $\sum_{n=1}^{n=\infty} \alpha_n < \infty$ and $\sum_{n=1}^{n=\infty} \beta_n < \infty$ such that

$$d(x_{n+1}, D) \le (1 + \alpha_n)d(x_n, D) + \beta_n. \tag{2.5}$$

Lemma 2.2. [14] Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three sequences of non negative real numbers satisfying the following conditions:

$$p_{n+1} \le (1 + q_n)p_n + r_n, n \ge 0, \sum_{n=0}^{\infty} q_n < \infty, \sum_{n=0}^{\infty} r_n < \infty.$$

Then
(1) $\lim_{n\to\infty} p_n$ *exists.*
(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, *then* $\lim_{n\to\infty} p_n = 0$.

III. MAIN RESULTS

Theorem 3.1. Let D be a closed convex subset of complete CAT(0) space X and T :→ D be a generalized asymptotically quasi-nonexpansive mapping with $\{\mu_n\}$, $\{\kappa_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \kappa_n < \infty$. Suppose $S_f(T)$ is closed. Let $\{x_n\}$ be an iteration as (1.5), then sequence $\{x_n\}$ is of monotone type (A) and monotone of type (B) with respect to F(T). Moreover, $\{x_n\}$ converges strongly to a fixed point p of mapping T if only if $\lim_{n\to\infty} \inf d(x_n, S_f(T)) = 0$, where $d(x, S_f(T)) = \inf_{p\in S_f(T)}\{d(x, p)\}$.

Proof. The necessary is obvious and so it is omitted. Now, we prove the sufficiency.

For $p \in S_f(T)$, from (1.5), (2.2) and definition 2.2, we have

$$\begin{aligned}
d(y_n, p) &= d((1-b_n)x_n \oplus b_n T^n x_n, p) \\
&\leq (1-b_n)d(x_n, p) + b_n d(T^n x_n, p) \\
&\leq (1-b_n)d(x_n, p) + b_n[d(1+\mu_n)d(x_n, p) + \kappa_n] \\
&\leq d(x_n, p) + b_n\mu_n d(x_n, p) + b_n\kappa_n \\
&\leq (1+b_n\mu_n)d(x_n, p) + b_n\kappa_n,
\end{aligned}$$

this give

$$d(y_n, p) \leq (1+b_n\mu_n)d(x_n, p) + b_n\kappa_n, \qquad (3.1)$$

$$\begin{aligned}
d(z_n, p) &= d((1-c_n)x_n \oplus c_n y_n, p) \\
&\leq (1-c_n)d(x_n, p) + c_n d(y_n, p) \\
&\leq (1-c_n)d(x_n, p) + c_n[(1+\mu_n b_n)d(x_n, p) + b_n\kappa_n] \\
&\leq d(x_n, p) + c_n\mu_n b_n d(x_n, p) + b_n c_n\kappa_n \\
&\leq (1+c_n\mu_n b_n)d(x_n, p) + b_n c_n\kappa_n,
\end{aligned}$$

this give

$$d(z_n, p) \leq (1+c_n\mu_n b_n)d(x_n, p) + b_n c_n\kappa_n. \qquad (3.2)$$

From (3.1) and (3.2), we have

$$\begin{aligned}
d(x_{n+1}, p) &= d((1-a_n)T^n z_n \oplus a_n T^n y_n, p) \\
&\leq (1-a_n)d(T^n z_n, p) + a_n d(T^n y_n, p) \\
&\leq (1-a_n)[(1+\mu_n)d(z_n, p) + \kappa_n] + a_n[(1+\mu_n)d(y_n, p) + \kappa_n] \\
&\leq (1-a_n)(1+\mu_n)d(z_n, p) + (1-a_n)\kappa_n + a_n(1+\mu_n)d(y_n, p) + a_n\kappa_n
\end{aligned}$$

$\leq (1-a_n)(1+\mu_n)[(1+c_n\mu_n b_n)d(x_n,p)+b_n c_n\kappa_n]+a_n(1+\mu_n)[(1+b_n\mu_n)d(x_n,p)+ b_n\kappa_n] + (1-a_n)\kappa_n + a_n\kappa_n$

$\leq (1-a_n)(1+\mu_n)(1+c_n\mu_n b_n)d(x_n,p) + (1-a_n)(1+\mu_n)b_n c_n\kappa_n + a_n(1+ \mu_n)(1+b_n\mu_n)d(x_n,p) + a_n(1+\mu_n)b_n\kappa_n + \kappa_n$

$\leq [(1-a_n)(1+\mu_n)(1+c_n\mu_n b_n)+a_n(1+\mu_n)(1+b_n\mu_n)]d(x_n,p)+(1-a_n)(1+ \mu_n)b_n c_n\kappa_n + a_n(1+\mu_n)b_n\kappa_n + \kappa_n$

$\leq (1+\mu_n)[(1-a_n)(1+c_n\mu_n b_n) + a_n(1+b_n\mu_n)]d(x_n,p) + (1-a_n)(1+ \mu_n)b_n c_n\kappa_n + a_n(1+\mu_n)b_n\kappa_n + \kappa_n$

$\leq (1+\mu_n)[1+b_n c_n\mu_n - a_n b_n c_n\mu_n + a_n b_n\mu_n]d(x_n,p)+(1-a_n)(1+\mu_n)b_n c_n\kappa_n + a_n(1+\mu_n)b_n\kappa_n + \kappa_n$

$\leq (1+b_n c_n\mu_n - a_n b_n c_n\mu_n + a_n b_n\mu_n + \mu_n + b_n c_n\mu_n^2 - a_n b_n c_n\mu_n^2 + a_n b_n\mu_n^2)d(x_n,p)+ (1-a_n)(1+\mu_n)b_n c_n\kappa_n + a_n(1+\mu_n)b_n\kappa_n + \kappa_n$

$= (1+P_n)d(x_n,p) + Q_n,$

this give

$$d(x_{n+1},p) \leq (1+P_n)d(x_n,p) + Q_n, \tag{3.3}$$

where $P_n = b_n c_n\mu_n - a_n b_n c_n\mu_n + a_n b_n\mu_n + \mu_n + b_n c_n\mu_n^2 - a_n b_n c_n\mu_n^2 + a_n b_n\mu_n^2$ and $Q_n = (1-a_n)(1+\mu_n)b_n c_n\kappa_n + a_n(1+\mu_n)b_n\kappa_n + \kappa_n$.

Sine by hypothesis, $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\kappa_n < \infty$, its follows that $\sum_{n=1}^{\infty}P_n < \infty$ and $\sum_{n=1}^{\infty}Q_n < \infty$.
Now from (3.3), we get

$$d(x_{n+1},S_f(T)) \leq (1+P_n)d(x_n,S_f(T)) + Q_n. \tag{3.4}$$

These inequalities, respectively, prove that $\{x_n\}$ is sequence of mono tone type (A) and monotone type (B) with respect to $S_f(T)$. Now, we prove that $\{x_n\}$ converges strongly to a fixed point of the mapping T if and only if lim $\inf_{n\to\infty} d(x_n, S_f(T)) = 0$.
If $x_n \to p \in S_f(T)$, then $\lim_{n\to\infty} d(x_n, p) = 0$.

Since $0 \leq d(x_n, S_f(T)) \leq d(x_n, p)$, we have lim $\inf_{n\to\infty} d(x_n, F(T)) = 0$ conversely, suppose that lim $\inf_{n\to\infty} d(x_n, F(T)) = 0$. Apply lemma 2.2 to (3.4), we have that lim $\inf_{n\to\infty} d(x_n, F(T))$ exists. Further by hypothesis lim $\inf_{n\to\infty} d(x_n F(T) = 0$, we conclude that lim $\inf_{n\to\infty} d(x_n, F(T)) = 0$,

$d(x_{n+m}, p) \leq (1 + P_{n+m-1})d(x_{n+m-1}, p) + Q_{n+m-1}$

$\leq e^{P_{n+m-1}} d(x_{n+m-1}, p) + Q_{n+m-1}$

$\leq e^{P_{n+m-1}}[e^{P_{n+m-1}} d(x_{n+m-2}, p) + Q_{n+m-2}] + Q_{n+m-1}$

$\leq e^{(}P_{n+m-1} + P_{n+m-2})d(x_{n+m-2}, p) + e^{P_{n+m-1}}[Q_{n+m-2} + Q_{n+m-1}]$

$\leq ...$

$\leq e^{\sum_{k=n}^{n+m-1} P_k} d(x_n, p) + e^{\sum_{k=n+1}^{n+m-1} P_k}(\sum_{k=n+1}^{n+m-1} Q_k)$

$\leq e^{\sum_{k=n}^{n+m-1} P_k} d(x_n, p) + e^{\sum_{k=n}^{n+m-1} P_k}(\sum_{k=n}^{n+m-1} Q_k),$

this give

$$d(x_{n+m}, p) \leq e^{\sum_{k=n}^{n+m-1} P_k} d(x_n, p) + e^{\sum_{k=n}^{n+m-1} P_k}(\sum_{k=n}^{n+m-1} Q_k). \qquad (3.5)$$

Let $M = e^{\sum_{k=n}^{n+m-1} P_k}$, then $0 < M < \infty$, and
$d(x_{n+m}, p) \leq Md(x_n p) + M(\sum_{k=n}^{n+m-1})Q_k$ for all natural numbers m, n and $p \in S_f(T)$. Since $\liminf_{n\to\infty} d(x_n, S_f(T)) = 0$,
therefore for every $\epsilon > 0$, there exists a natural number N such that $d(x_n, S_f(T)) < \frac{\epsilon}{8M}$ and $(\sum_{k=n}^{n+m-1})Q_k < \frac{\epsilon}{4M}$, for all $n \geq N$. So we can find $p^* \in S_f(T)$ such that $d(x_N, p^*) < \frac{\epsilon}{4M}$. Hence, for all $n \geq N$, and $m \geq 1$, we have

$d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(x_n, p^*)$
$\leq Md(x_N, p^*) + M\sum_{k=N}^{\infty} Q_n + Md(x_N, p^*) + M\sum_{k=N}^{\infty} Q_n$
$\leq 2Md(x_N, p^*) + 2M\sum_{k=N}^{\infty} Q_n$
$\leq 2M\left(\frac{\epsilon}{4M} + \frac{\epsilon}{4M}\right) = \epsilon,$
this give

$$d(x_{n+m}, x_n) \leq \epsilon. \qquad (3.6)$$

This proves that $\{x_n\}$ is Cauchy sequence. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = z$. Since D is closed, therefore $z \in D$. Next, we show that $z \in S_f(T)$. Now, the following two inequalities:

$$d(z, p) \leq d(z, x_n) + d(x_n, p) \text{ for every } p \in S_f(T), n \in \mathbb{N}, \qquad (3.7)$$

equation (3.7) give

$$-d(z, x_n) \leq d(z, S_f(T)) - d(x_n, S_f(T)) \leq d(z, x_n), \forall p \in S_f(T), n \in \mathbb{N}, \qquad (3.8)$$

this gives

$$|d(z, S_f(T)) - d(x_n, S_f(T))| \leq d(z, x_n), n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} x_n = z$, and $\lim_{n\to\infty} d(x_n, S_f(T)) = 0$, these gives $z \in S_f(T)$. This completes the proof. □

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Lemma 3.2. *Let (X, d) be a complete $CAT(0)$ space, and let D be a nonempty closed, convex subset of X. Let $T : D \to D$ be a uniformly continuous generalized asymptotically quasi-nonexpansive mapping with $\{\mu_n\}, \{\kappa_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \kappa_n < \infty$. Suppose $S_f(T) \neq \emptyset$. Let $\{x_n\}$ be an iteration as (1.5). Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[\delta, 1-\delta]$ for $\delta \in (0, 1)$. If $\liminf_{n\to\infty} d(x_n, p) \leq \liminf_{n\to\infty} d(y_n, p)$. Then*
(a) $\lim_{n\to\infty} d(T^n z_n, T^n y_n) = 0$
(b) $\lim_{n\to\infty} d(x_n, T^n x_n) = 0.$

Proof. Let $p \in S_f(T)$. Then by Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists. Let $\lim_{n\to\infty} d(x_n, p) = \alpha$. This give

$$\liminf_{n\to\infty} d(x_n, p) = \alpha \leq \liminf_{n\to\infty} d(y_n, p). \tag{3.9}$$

If $\alpha = 0$ then by continuity of T the conclusion follows.
Now if $\alpha > 0$.
We claim that $\lim_{n\to\infty} d(T^n z_n, T^n y_n) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n, T^n x_n) = 0$.
Since $\{x_n\}$ is bounded, then there exists $R > 0$ such that $\{x_n\}, \{y_n\}, \{z_n\} \subset N_R(p)$ for all $n \in \mathbb{N}$. Using (1.5) and (2.3), we have
$d^2(y_n, p) = d^2((1-b_n)x_n \oplus b_n T^n x_n, p)$
$\leq (1-b_n)d^2(x_n, p) + b_n d^2(T^n x_n, p) - b_n(1-b_n)d^2(x_n, T^n x_n)$
$\leq (1-b_n)d^2(x_n, p) + b_n[(1+\mu_n)d^2(x_n, p) + \kappa_n]^2 - b_n(1-b_n)d^2(x_n, T^n x_n)$
$\leq (1-b_n)d^2(x_n, p) + b_n(1+\mu_n)^2 d^2(x_n, p) + b_n\kappa_n^2 + b_n\kappa_n(1+\mu_n) - b_n(1-b_n)d^2(x_n, T^n x_n)$
$\leq (1-b_n)d^2(x_n, p) + b_n(1+\mu_n)^2 d^2(x_n, p) + \rho_n - b_n(1-b_n)d^2(x_n, T^n x_n)$
$\leq (1 + b_n\mu_n^2 + 2b_n\mu_n)d^2(x_n, p) + \rho_n - b_n(1-b_n)d^2(x_n, T^n x_n)$

$\leq (1+\tau_n)d^2(x_n, p) + \rho_n - b_n(1-b_n)d^2(x_n, T^n x_n)$,
this give

$$d^2(y_n, p) \leq (1+\tau_n)d^2(x_n, p) + \rho_n - b_n(1-b_n)d^2(x_n, T^n x_n), \tag{3.10}$$

equation (3.10) give

$$d^2(y_n, p) \leq (1+\tau_n)d^2(x_n, p) + \rho_n, \tag{3.11}$$

where $\rho_n = b_n\kappa_n^2 + b_n\kappa_n(1+\mu_n)$, $\tau_n = b_n\mu_n^2 + 2b_n\mu_n$.

$d^2(z_n, p) = d^2((1-c_n)x_n \oplus c_n y_n, p)$
$\leq (1-c_n)d^2(x_n, p) + c_n d^2(y_n, p) - c_n(1-c_n)d^2(x_n, y_n)$
$\leq (1-c_n)d^2(x_n, p) + c_n[(1+\tau_n)d^2(x_n, p) + \rho_n] - c_n(1-c_n)d^2(x_n, y_n)$
$\leq (1+c_n\tau_n)d^2(x_n, p) + c_n\rho_n - c_n(1-c_n)d^2(x_n, y_n)$
$\leq (1+\tau_n)d^2(x_n, p) + \rho_n - c_n(1-c_n)d^2(x_n, y_n)$,
which give

$$d^2(z_n, p) \leq (1+\tau_n)d^2(x_n, p) + \rho_n - c_n(1-c_n)d^2(x_n, y_n), \tag{3.12}$$

equation (3.12) give

$$d^2(z_n, p) \leq (1+\tau_n)d^2(x_n, p) + \rho_n. \tag{3.13}$$

Since by assumption that $\sum_{n=1}^{\infty} \mu_n < \infty$, and $\sum_{n=1}^{\infty} \kappa_n < \infty$, it follows that $\sum_{n=1}^{\infty} \tau_n < \infty$, and $\sum_{n=1}^{\infty} \rho_n < \infty$.

Again note that

$$\begin{aligned}
&d^2(x_{n+1}, p) = d^2((1-a_n)T^n z_n \oplus a_n T^n y_n, p) \\
&\leq (1-a_n)d^2(T^n z_n, p) + a_n d^2(T^n y_n, p) - a_n(1-a_n)d^2(T^n z_n, T^n y_n) \\
&\leq (1-a_n)[(1+\mu_n)^2 d(z_n, p) + \kappa_n^2 + 2\kappa_n(1+\mu_n)d(z_n, p)] + a_n[(1+\mu_n)^2 d^2(y_n, p) + \\
&\kappa_n^2 + 2\kappa_n(1+\mu_n)d(y_n, p)] - a_n(1-a_n)d^2(T^n z_n, T^n y_n) \\
&\leq (1-a_n)[(1+\mu_n)^2 d^2(z_n, p) + \alpha_n] + a_n[(1+\mu_n)^2 d^2(y_n, p) + \beta_n] - a_n(1- \\
&a_n)d^2(T^n z_n, T^n y_n) \\
&\leq (1-a_n)[(1+\mu_n)^2((1+\tau_n)d^2(x_n, p) + \rho_n) + \alpha_n] + a_n[(1+\mu_n)^2((1+ \\
&\tau_n)d^2(x_n, p) + \rho_n) + \beta_n] - a_n(1-a_n)d^2(T^n z_n, T^n y_n) \\
&\leq (1+\mu_n)^2(1+\tau_n)d^2(x_n, p) + \rho_n + (1-a_n)\alpha_n + a_n\beta_n - a_n(1-a_n)d^2(T^n z_n, T^n y_n) \\
&\leq (1+t_n)d^2(x_n, p) + \phi_n - a_n(1-a_n)d^2(T^n z_n, T^n y_n)
\end{aligned}$$

Which give

$$d^2(x_{n+1}, p) \leq (1+t_n)d^2(x_n, p) + \phi_n - a_n(1-a_n)d^2(T^n z_n, T^n y_n), \quad (3.14)$$

equation (3.14) give

$$d^2(x_{n+1}, p) \leq (1+t_n)d^2(x_n, p) + \phi_n, \quad (3.15)$$

where $\alpha_n = \kappa_n^2 + 2\kappa_n(1+\mu_n)d(z_n, p)$, $\beta_n = \kappa_n^2 + 2\kappa_n(1+\mu_n)d(y_n, p)$, $t_n = \tau_n\mu_n^2 + 2\tau_n\mu_n + \mu_n^2 + 2\mu_n$ and $\phi_n = \rho_n + (1-a_n)\alpha_n + a_n\beta_n$. Since $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \kappa_n < \infty, \sum_{n=1}^{\infty} \tau_n < \infty$, and $\sum_{n=1}^{\infty} \rho_n < \infty$ it follows that $t_n < \infty$, and $\sum_{n=1}^{\infty} \phi_n < \infty$. Observe that $\delta^2 \leq a_n(1-a_n)$ and $\sum_{n=1}^{\infty} t_n < \infty$ and $\sum_{n=1}^{\infty} \phi_n < \infty$. For $m \geq 1$, (3.14) implies

$$\begin{aligned}
&\sum_{n=1}^{m} d^2(T^n z_n, T^n y_n) \leq \frac{1}{\delta}\left[d^2(x_1, p) - d^2(x_{m+1}, p) + \sum_{n=1}^{m} t_n d^2(x_n, p) + \sum_{n=1}^{m} \phi_n\right] \\
&\leq \frac{1}{\delta}\left[d^2(x_1, p) + R^2 \sum_{n=1}^{m} t_n + \sum_{n=1}^{m} \phi_n\right]
\end{aligned}$$

when $m \to \infty$, we have $\sum_{n=1}^{\infty} d^2(T^n z_n, T^n y_n) < \infty$, since $\sum_{n=1}^{\infty} t_n, \sum_{n=1}^{\infty} \phi_n < \infty$ ad $d(x_n, p) \leq R \; \forall \; n \geq 1$.

Hence

$$\lim_{n\to\infty} d(T^n z_n, T^n y_n) = 0. \quad (3.16)$$

Thus assertion (a) of the lemma is proved.

$d(y_n, p) = d((1-b_n)x_n \oplus b_n T^n x_n, p)$
$\leq (1-b_n)d(x_n, p) + b_n d(T^n x_n, p)$
$\leq (1-b_n)d(x_n, p) + b_n[(1+\mu_n)d(x_n, p) + \kappa_n]$
$\leq (1-b_n\mu_n)d(x_n, p) + b_n\kappa_n,$
from which we deduce that $\limsup_{n\to\infty} d(y_n, p) \leq \alpha$. Therefore $\lim\limits_{n\to\infty} d(y_n, p) = \alpha$.
Now from (3.10), for $m \geq 1$, implies

for $m \geq 1$, implies

$\sum_{n=1}^{m} d^2(x_n, T^n y_n) \leq \frac{1}{\delta^2}[\sum_{n=1}^{m} \tau_n d^2(x_n, p) + \sum_{n=1}^{m} \rho_n]$

$\leq \frac{1}{\delta^2}[R^2 \sum_{n=1}^{m} \tau_n + \sum_{n=1}^{m} \rho_n]$
when $m \to \infty$, we have
$\sum_{n=1}^{\infty} d^2(x_n, T^n y_n) < \infty$, since $\sum_{n=1}^{\infty} \tau_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and $d(x_n, p) \leq R \ \forall\ n \geq 1$.

Hence

$$\lim_{n\to\infty} d(x_n, T^n y_n) = 0. \tag{3.17}$$

This assertion (b) of the Lemma is proved. This completes the proof of Lemma 3.2.

□

Lemma 3.3. *Let (X, d) be a complete $CAT(0)$ space, and let D be a nonempty closed, convex subset of X. Let $T : D \to D$ be a uniformly 1-Lipschitzian generalized asymptotically quasi-nonexpansive mapping with $\{\mu_n\}, \{\kappa_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \kappa_n < \infty$. Suppose $S_f(T) \neq \emptyset$. Let $\{x_n\}$ be an iteration as (1.5). Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[\delta, 1-\delta]$ for $\delta \in (0, 1)$. If $\liminf_{n\to\infty} d(x_n, p) \leq \liminf_{n\to\infty} d(y_n, p)$. Then $\lim\limits_{n\to\infty} (Tx_n, x_n) = 0$.*

Proof. From Lemma 3.2, we have

$$\lim_{n\to\infty} d(T^n z_n, T^n y_n) = 0 \text{ and } \lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{3.18}$$

From (3.18) and (1.5), we have

$$\lim_{n\to\infty} d(y_n, x_n) = 0 \text{ and } \lim_{n\to\infty} d(z_n, x_n) = 0. \tag{3.19}$$

Since T is uniformly 1-Lipschitzian, we have

$$d(x_{n+1}, x_n) = d((1-a_n)T^n z_n \oplus a_n T^n y_n, x_n)$$

$$\le (1-a_n)d(T^n z_n, x_n) + a_n d(T^n y_n, x_n)$$

$$\le (1-a_n)[d(T^n z_n, T^n x_n) + d(T^n x_n, x_n)] + a_n[d(T^n y_n, T^n x_n) + d(T^n x_n, x_n)]$$

$$\le (1-a_n)[d(z_n, x_n) + d(T^n x_n, x_n)] + a_n[d(y_n, x_n) + d(T^n x_n, x_n)]$$

$$\le (1-a_n)d(z_n, x_n) + a_n d(y_n, x_n) + d(T^n x_n, x_n),$$

this give

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.20}$$

Again since T is uniformly 1-Lipschitziann, equations (3.20) and (3.18), we have

$$d(x_{n+1}, T^n x_{n+1}) \le d(x_{n+1}, x_n) + d(T^n x_n, x_n) + d(T^n x_{n+1}, T^n x_n)$$
$$\le d(x_{n+1}, x_n) + d(T^n x_n, x_n) + d(x_{n+1}, x_n)$$
$$\le 2d(x_{n+1}, x_n) + d(T^n x_n, x_n),$$

which give

$$\lim_{n\to\infty} d(x_{n+1}, T^n x_{n+1}) = 0. \tag{3.21}$$

Since T is uniformly 1-Lipschitzian, we get
$d(x_{n+1}, Tx_{n+1}) \le d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, Tx_{n+1})$
$\le d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^n x_{n+1}, x_{n+1}),$
which give,

$$\lim_{n\to\infty} d(x_{n+1}, Tx_{n+1}) = 0, \tag{3.22}$$

which give,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.23}$$

This completes the proof. □

Theorem 3.4. *Let (X, d) be a complete $CAT(0)$ space, and let D be a nonempty closed, convex subset of X. Let $T : D \to D$ be a uniformly 1-Lipschitzian generalized asymptotically quasi-nonexpansive mapping with $\{\mu_n\}, \{\kappa_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \kappa_n < \infty$. Suppose $S_f(T) \neq \emptyset$. Let $\{x_n\}$ be an iteration as (1.5). Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be real sequences in $[\delta, 1-\delta]$ for $\delta \in (0, 1)$. Assume, in addition that T is compact. Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

Proof. By Lemma 3.3, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0 and \lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.23}$$

$\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$.
Again by Theorem 3.1, $\{x_n\}$ is bounded. It follows by our assumption T is compact, then there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to x^* \in D$ as $k \to \infty$. By again, we have

$$d(x^*, Tx^*) = \lim_{k\to\infty} d(Tx_{n_k}, x_{n_k}) = 0. \tag{1}$$

It shows that $x^* \in S_f(T)$. Furthermore, since $\lim_{n\to\infty} d(x_n, x^*)$ exists, therfore $\lim_{n\to\infty} d(x_n, x^*) = 0$ that is, x_n is converges to a fixed point of T. □

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