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Using power series to find solution of homogeneous linear differential equations

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ABSTRACT: The laws of the universe are written in the language of mathematics. Algebra is enough to solve many static problems. However, Most of the natural phenomena of interest involve variation and are often described by equations that involve quantitative change, that is differential equations. Especially, linear differential equations appear in many physics problems such as Vibrating Springs, Damped Vibrations, Forced Vibrations, Electric Circuits. Therefore, developing methods to solve differential equations is very important in analysis.

KEYWORDS: Power series, solution, differential equation, homogeneous linear.

I. INTRODUCTION

The method of solving the homogeneous first-order linear differential equations and the second-order homogeneous linear differential equations are presented in [3]. So it is easy to solve the first- order linear differential equations or the second- order linear differential equations with constant coefficients. To find the general solution of a the second- order homogeneous linear differential equation with function coefficients, we need to find its two linear independent eigensolutions, this is difficult for some equations, so it is necessary to develop other methods to find solutions to these equations. Using power series to find solution of homogeneous linear differential equations is an effective method to solve this problem.

II. METHOD USING POWER SERIES

To find the power series solution of a homogeneous linear differential equation form: $y=\sum_{n=0}^{\infty} a_n x^n$ (1)

It is notice that the following property of the power series: In the convergence interval of the power series, it can take the derivative and integrate each term of the series, the new series has the same radius of convergence as the original series. Therefore, formula (1) is takeen the derivative and then substitute it in the differential equation and from there determine the values of the constants so that it correctly solves the differential equation. First, to illustrate this method, consider the following example.

Example 1. Solve the differential equation: $y''-y=0$ (2)

Solution. Assume $y=\sum_{n=0}^{\infty} a_n x^n$ (3)

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

We reindex the series (replace n with n+2) to obtain

$$y''=\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n}$$

Putting y’’ and y into Equation (2)

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_n\right] x^{n} = 0 \quad (4)$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore the coeffcients of x^n in Equation (4) must be 0:

$$(n+2)(n+1)a_{n+2}-a_n=0$$

$$\Leftrightarrow a_{n+2}=\frac{a_n}{(n+1)(n+2)}$$

Put $n=0 \Rightarrow a_2=\frac{a_0}{1.2}=\frac{a_0}{2!}$

Put $n=1 \Rightarrow a_3 = \frac{a_1}{2.3} = \frac{a_1}{3!}$

Put $n=2 \Rightarrow a_4 = \frac{a_2}{3.4} = \frac{a_0}{1.2.3.4} = \frac{a_0}{4!}$

Put $n=3 \Rightarrow a_5 = \frac{a_3}{4.5} = \frac{a_1}{2.3.4.5} = \frac{a_1}{5!}$

For the even coefficients: $a_{2n} = \frac{a_0}{(2n)!}$

For the odd coefficients: $a_{2n+1} = \frac{a_1}{(2n+1)!}$

Putting these values back into Equation (3), the solution is written as

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + ...$$

$$= a_0 + a_1 x + \frac{a_0}{2!}x^2 + \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + ...$$

$$= a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)$$

Maclaurin series of e^x and e^{-x} are known as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ... \quad (5)$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + ... \quad (6)$$

Adding each side (5) and (6):

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$

Subtracting each side (5) and (6):

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$

Thus, the solution is

$$y = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)$$

$$= a_0 . \frac{e^x + e^{-x}}{2} + a_1 \frac{e^x - e^{-x}}{2}$$

$$= \frac{a_0 + a_1}{2} e^x + \frac{a_0 - a_1}{2} e^{-x}$$

$$= C_1 e^x + C_2 e^{-x}$$

(for $C_1 = \frac{a_0 + a_1}{2}, C_2 = \frac{a_0 - a_1}{2}$)

The solution of equation (2) is found in a simple way by the method of solving the characteristic equation $k^2 - 1 = 0 \Leftrightarrow k = \pm 1$ and solution is $y = C_1 e^x + C_2 e^{-x}$

The method using power series is also applicable to first- order homogeneous linear differential equations.

Example 2. Solve the differential equation: $y' - xy = 0$ (7)

Solution. Asume that $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Putting y'and y into Equation (7):

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Leftrightarrow a_1 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} - a_{n-1}\right] x^n = 0 \quad (8)$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore the coeffcients of x^n and a_1 in Equation (8) must be 0:

$$\begin{cases} a_1 = 0 \\ a_{n+1} = \frac{a_{n-1}}{n+1} \end{cases}$$

Put $n=1 \Rightarrow a_2 = \frac{a_0}{2}$

Put $n=2 \Rightarrow a_3 = \frac{a_1}{3} = 0$

Put $n=3 \Rightarrow a_4 = \frac{a_2}{4} = \frac{a_0}{2.4} = \frac{a_0}{2^2.2!}$

Put $n=4 \Rightarrow a_5 = \frac{a_3}{3} = 0$

Put $n=5 \Rightarrow a_6 = \frac{a_4}{6} = \frac{a_0}{2.4.6} = \frac{a_0}{2^3.3!}$

For the even coefficients, $a_{2n} = \frac{a_0}{2^n.n!}$

For the odd coefficients, $a_{2n+1} = 0$

Thus, solution of equation is

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + ...$$

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$$= a_0 + \frac{a_0}{2}x^2 + \frac{a_0}{2^2.2!}x^4 + \frac{a_0}{2^3.3!}x^6 + ...$$

$$= a_0\left(1 + \frac{x^2}{2^1.1!} + \frac{(x^2)^2}{2^2.2!} + \frac{(x^2)^3}{2^3.3!} + ...\right)$$

It is recognized the series obtained in solution above as being the Maclaurin series for $e^{\frac{x^2}{2}}$. Therefore the solution is written as $y = a_0 e^{\frac{x^2}{2}}$.

These are simple examples illustrating how to use power series to find solutions of homogeneous linear differential equations. To see the effectiveness of this method, consider the following examples.

Example 3. Solve the differential equation: $y'' - xy' - y = 0$ (9)

Solution. Suppose $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Putting y'', y' and y into Equation (9):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n a_n - a_n\right]x^n = 0$$

Thus, we have $a_{n+2} = \frac{(n+1)a_n}{(n+2)(n+1)} = \frac{a_n}{n+2}$

$$\Rightarrow \begin{cases} a_2 = \frac{a_0}{2}, a_4 = \frac{a_0}{2.4}, a_6 = \frac{a_0}{2.4.6}, ... \\ a_3 = \frac{a_1}{3}, a_5 = \frac{a_1}{3.5}, a_7 = \frac{a_1}{3.5.7}, ... \end{cases}$$

Thus, the solution is

$$y = a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{2.4}x^4 + \frac{a_1}{3.5}x^5 + ...$$

$$= a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{2.4} + ...\right) + a_1\left(x + \frac{x^3}{3} + \frac{x^5}{3.5} + ...\right)$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

We recognize $y_1(x)$ is the Maclaurin series for $e^{\frac{x^2}{2}}$. However, $y_2(x)$ do not define any elementary functions. Therefor, if this equation is solved by the method of finding 2 linear independent eigen solutions, it will be very difficult to find eigensolutions $y_2(x)$.

The power series method is also used to solve the initial-value problems.

Example 4. Find solution of the differential equation: $(1+x^2)y'' - 4xy' + 6y = 0$ (10), $y(0) = 0,\ y'(0) = 1$

Solution. Suppose $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Putting y'', y' and y into Equation (10):

$$(1+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n - 4\sum_{n=1}^{\infty} n a_n x^n$$

$$+6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 4n a_n x^n$$

$$+\sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n\right]x^n = 0$$

Thus, the coeffcients of x^n must be 0, we get:

$$a_{n+2} = -\frac{(n^2 - 5n + 6)a_n}{(n+2)(n+1)} = -\frac{(n-2)(n-3)a_n}{(n+2)(n+1)} \quad (11)$$

One of the given conditions is

$y(0) = a_0 + a_1.0 + a_2.0 + ... \Leftrightarrow a_0 = 0$

The other given condition is

$y'(0) = a_1 + a_2.0 + a_3.0 + ... \Leftrightarrow a_1 = 1$

Since the recursion formula (11), we have

$n = 0 \Rightarrow a_2 = -\frac{6a_0}{2} = 0$

$n = 1 \Rightarrow a_3 = -\frac{a_1}{3}$

$n = 2 \Rightarrow a_4 = 0$

$n = 3 \Rightarrow a_5 = 0 \Rightarrow a_n = 0 \ \forall n \geq 2$

Therefore the solution of the initial-value problem is

$y = x - \frac{1}{3}x^3$

III. CONCLUSIONS

There are many methods to solve differential equations, but the method using power series will be especially effective when solving homogeneous linear differential equations with function coefficients which is very difficult or even impossible to find two independent eigensolutions.

Conflict of interest

The author declares no conflict of interest.

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