

Weibull-Exponential Power distribution and its application

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We develop a new model named the Weibull-Exponential Power distribution (WEPD) which extends the two-parameter Exponential Power distribution (EPD) and has increasing, decreasing, and bathtub shapes for the hazard rate function. Various structural statistical properties of the WEPD are studied including explicit expressions for the hazard and quantile functions, ordinary moments and central moments, Bonferroni and Lorenz curves, mean residual life, mean waiting time, mean deviations, probability weighted moments, generating function and order statistics. The measures of uncertainty (Renyi and q entropies) are also considered. The model parameters are determined using the method of maximum likelihood technique and the observed information matrix. The capability of the new model is illustrated by means of two real life data sets. Based on these data, the new model performs better than exponential power,

I. Introduction

The flexibility of classical distributions, such as Weibull, gamma, and exponential, has been a significant focus of research in recent decades. Several studies have introduced new parameters into these distributions to improve their modeling capabilities and adapt them to various types of data. A notable example is the approach proposed by Mudholkar and Srivastava (1993), which adds an extra parameter to the Weibull distribution, allowing it to handle a bathtub-shaped failure rate function (hrf). In this context, several other studies have made substantial contributions, including those by Marshall and Olkin (1997), Gupta et al. (1998), Eugene et al. (2002), Zografos and Balakrishnan (2009), Cordeiro and de Castro (2011), Alexander et al. (2012), Cordeiro et al. (2013), Alzaatreh et al. (2013), Bourguignon et al. (2014), Alizadeh et al. (2015), Chipepa et al. (2019), Baharith and Alamoudi (2021), and Tlhaloganyang et al. (2022), among many others

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II. Weibull Exponential Power distribution and Applications to

Smith and Bain (1975) developed exponential power distribution (EPD) with the CDF and PDF expressed as follows

$$G(x; \varsigma, \vartheta) = 1 - \exp \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^\vartheta \right) \right), \quad (1)$$

and

$$g(x; \varsigma, \vartheta) = \frac{\vartheta}{\varsigma} \left(\frac{x}{\varsigma} \right)^{\vartheta-1} \exp \left(\left(\frac{x}{\varsigma} \right)^\vartheta \right) \exp \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^\vartheta \right) \right), \quad x > 0; \varsigma, \vartheta > 0 \quad (2)$$

Respectively, where ς is a scale parameter and ϑ is a shape parameter. Using a flexible generator Weibull-G generator developed by Bourguignon et al. (2014), with CDF given by

$$F(x) = 1 - \exp \left(-\delta \left[\frac{G(x)}{1 - G(x)} \right]^\vartheta \right). \quad (3)$$

Inserting (1) into (3), we obtain the CDF of the WEPD given as

$$F(x) = 1 - \exp \left(-\delta \left[1 - \exp \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^\vartheta \right) \right) \right]^{-\vartheta} \right), \quad (4)$$

and the PDF associated with (4) is

$$f(x) = \delta \vartheta \frac{1}{\varsigma} \left(\frac{x}{\varsigma} \right)^{\vartheta-1} \exp \left(\left(\frac{x}{\varsigma} \right)^\vartheta \right) \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^\vartheta \right) \right) \exp \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^\vartheta \right) \right)$$

$$\times \left\{ 1 - \exp \left(1 - \exp \left(\left(\frac{x}{\zeta} \right)^{\vartheta} \right) \right) \right\}^{-\theta-1} \exp \left(-\delta \left[1 - \exp \left(1 - \exp \left(\left(\frac{x}{\zeta} \right)^{\vartheta} \right) \right) \right]^{-\theta} \right) \quad (5)$$

And expression for the reliability and the hazard function is given by

$$R(x) = \exp \left(-\delta \left[1 - \exp \left(1 - \exp \left(\left(\frac{x}{\zeta} \right)^{\vartheta} \right) \right) \right]^{-\theta} \right) \quad (6)$$

and

$$h(x) = \delta \theta \vartheta \frac{1}{\zeta} \left(\frac{x}{\zeta} \right)^{\vartheta-1} \exp \left(\left(\frac{x}{\zeta} \right)^{\vartheta} \right) \left(1 - \exp \left(\left(\frac{x}{\zeta} \right)^{\vartheta} \right) \right) \exp \left(1 - \exp \left(\left(\frac{x}{\zeta} \right)^{\vartheta} \right) \right) \times \left\{ 1 - \exp \left(1 - \exp \left(\left(\frac{x}{\zeta} \right)^{\vartheta} \right) \right) \right\}^{-\theta-1} \quad (7)$$

Where ϑ and θ are positive shape parameters and ζ is a positive scale parameter. Figure 1 is the graphs of the CDF and the PDF of the WEPD plotted for the hypothetical values of the parameters. Figure 1b shows that the shape of the probability density function is uni-modal and is positively skewed for different values of the parameters. Figure 2 is the graph of the hazard and

the survival functions. 2(a) shows that as time increases the survival probability of an items decreases with time and 2(b) indicates that the WEPD can be used to model data exhibiting various shapes of the hazard rate functions ranging from decreasing, increasing, decreasing-increasing, increasing-decreasing and the bathtub shapes.

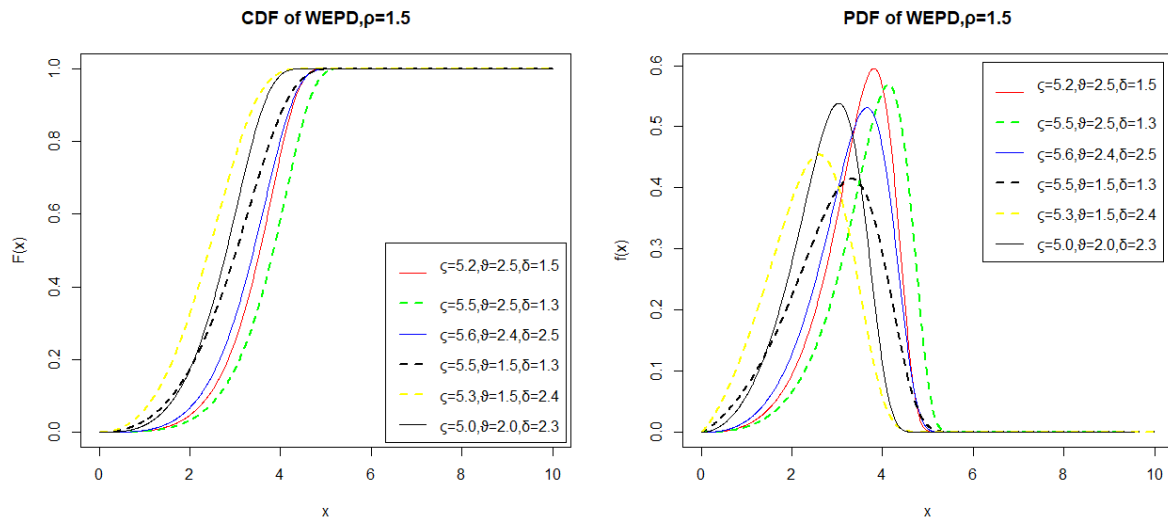


Figure 1. plots of the CDF (a) and the PDF (b) functions of the WEPD

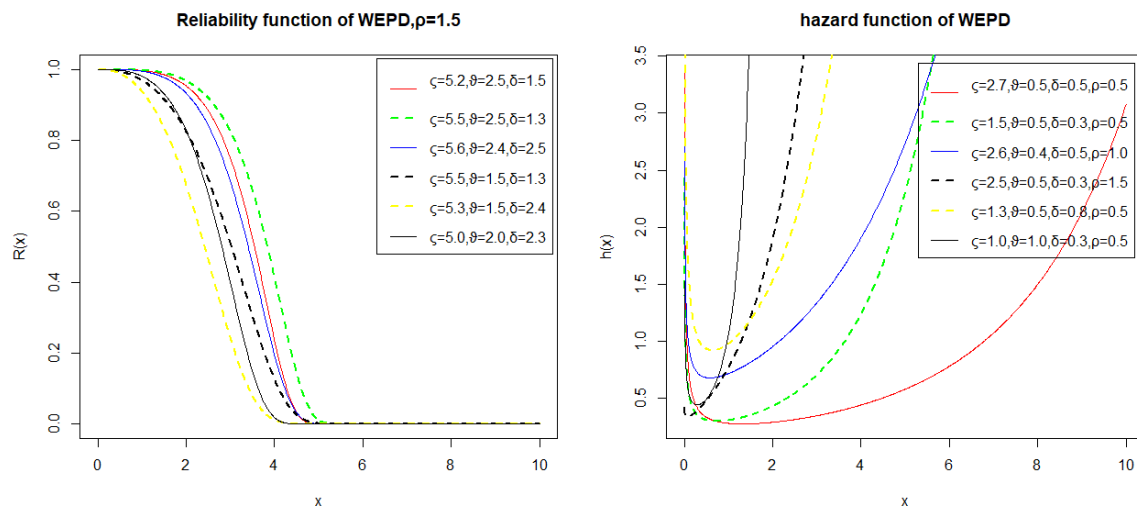


Figure 2. plots of the survival (a) and the hazard (b) functions of the WEPD

2.1 Quantile function

Here, we obtain the qf of the WEPD. The qf is derived as follows. Suppose u represent a random variable obtained from a standard uniform distribution and $F(x; \zeta, \vartheta, \theta) = u$. The qf of the WEPD can be derived via solving for x in the giving equation

$$u = 1 - \exp\left(-\delta \left[1 - \exp\left(\left(\frac{x}{\zeta}\right)^{\vartheta}\right)^{-\theta}\right]\right).$$

Based on this, finally we have

$$x_u = \zeta \left(\log \left[1 - \log \left(1 - \left\{ -\frac{1}{\delta} [\log(1-u)] \right\}^{1/\theta} \right) \right] \right)^{1/\vartheta} \quad (8)$$

III. Moments

The r^{th} moment of the WEPD random variable X is expressed as

$$E(X^s) = \mu'_s = \delta \vartheta \theta \zeta^{-\vartheta} \sum_{r=0}^{\infty} \mathcal{Z}_r \int_{-\infty}^{\infty} x^{\frac{s+\vartheta-1}{\vartheta}} \exp\left((r+1)\left(\frac{x}{\zeta}\right)^{\vartheta}\right) \quad (9)$$

The closed form expression for the moment of WEPD given by

$$\mu'_1 = \delta \vartheta \theta \sum_{r=0}^{\infty} \mathcal{Z}_r \zeta^{-s} (-1)^{s+\vartheta-1} (r+1)^{-\left(\frac{s}{\vartheta}+1\right)} \Gamma\left(\frac{s}{\vartheta}+1\right) \quad (10)$$

And for $s = 1$, the mean is obtained as

$$\mu'_1 = \delta \vartheta \theta \sum_{r=0}^{\infty} \mathcal{Z}_r \zeta^{-1} (-1)^{\vartheta} (r+1)^{-\left(\frac{1}{\vartheta}+1\right)} \Gamma\left(\frac{1}{\vartheta}+1\right) \quad (11)$$

3.1 Renyi Entropy

The Rényi entropy was introduced by Renyi (1961), and it is mathematically represented by

$$R_{\lambda}(X) = \frac{1}{1-\lambda} \log\{Z(\lambda)\} \quad (12)$$

$Z(\lambda) = \int_{-\infty}^{\infty} f^{\lambda}(x) dx$, $\lambda > 0$, and $\lambda \neq 1$. Using the (5), followed by applying binomial expansion, we have

$$Z(\lambda) = \left(\frac{\delta \theta \zeta}{\vartheta}\right)^{\lambda} \sum_{l=0}^{\infty} W_l \int_{-\infty}^{\infty} x^{(\theta-1)\lambda} \exp\left[\left(1+l\right)\left(\frac{x}{\zeta}\right)^{\vartheta}\right] dx \quad (15)$$

where

$$W_l = \sum_{i=j=k=0}^{\infty} \frac{(-1)^l}{i! k!} \binom{\theta(1+i)+j}{\lambda} \delta^i \lambda^i (\lambda+j)^k \binom{k+\lambda}{l}$$

Then, we have

$$Z(\lambda) = \varsigma^{(\theta-1)\lambda+1} \vartheta^{\lambda-1} \delta^{\lambda} \theta^{\lambda} (1+l)^{-\frac{\lambda(\theta-1)+1}{\vartheta}} \Gamma\left(\frac{\lambda(\theta-1)+1}{\vartheta}\right) \quad (16)$$

Combining (14), and (16), we derive an expression for the Rényi entropy of the WEPD as

$$R_{\lambda}(X) = \frac{1}{1-\lambda} \log \left\{ \sum_{l=0}^{\infty} W_l \varsigma^{(\theta-1)\lambda+1} \vartheta^{\lambda-1} \delta^{\lambda} \theta^{\lambda} (1+l)^{-\frac{\lambda(\theta-1)+1}{\vartheta}} \Gamma\left(\frac{\lambda(\theta-1)+1}{\vartheta}\right) \right\} \quad (17)$$

3.2 Maximum likelihood estimate

$$\begin{aligned} l(x) = & n \log \left(\delta \theta \vartheta \frac{1}{\varsigma} \right) + (\vartheta - 1) \sum_{i=1}^n \left(\frac{x}{\varsigma} \right) + \sum_{i=1}^n \left(\frac{x}{\varsigma} \right)^{\vartheta} + \sum_{i=1}^n \log \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^{\vartheta} \right) \right) \\ & + \sum_{i=1}^n \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^{\vartheta} \right) \right) - (\theta + 1) \sum_{i=1}^n \log \left\{ 1 - \exp \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^{\vartheta} \right) \right) \right\} \\ & + \sum_{i=1}^n \left(-\delta \left[1 - \exp \left(1 - \exp \left(\left(\frac{x}{\varsigma} \right)^{\vartheta} \right) \right) \right]^{-\theta} \right) \end{aligned} \quad (18)$$

IV. Application

We estimate the unknown parameters of each model by maximum likelihood using L-BFGS-B method and the goodness-of-fit statistics Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Cramér-von Mises (W), and P values (P) are used to compare the models. The minimum the values of AIC, BIC, CAIC, HQIC, K, CV, and higher the P values the better the fit of the model to the data. The data represents the fatigue fracture life of Kevlar 373/epoxy subjected to constant pressure at the 90-stress level until all specimens failed. The observations are as follows: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751,

0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960. Figure 3 displays the Violin plot, Total Time on Test (TTT) plot, and kernel density plot. The TTT plot indicates that the data exhibits non-monotone failure rate, and the kernel density plot shows that the data is positively skewed.

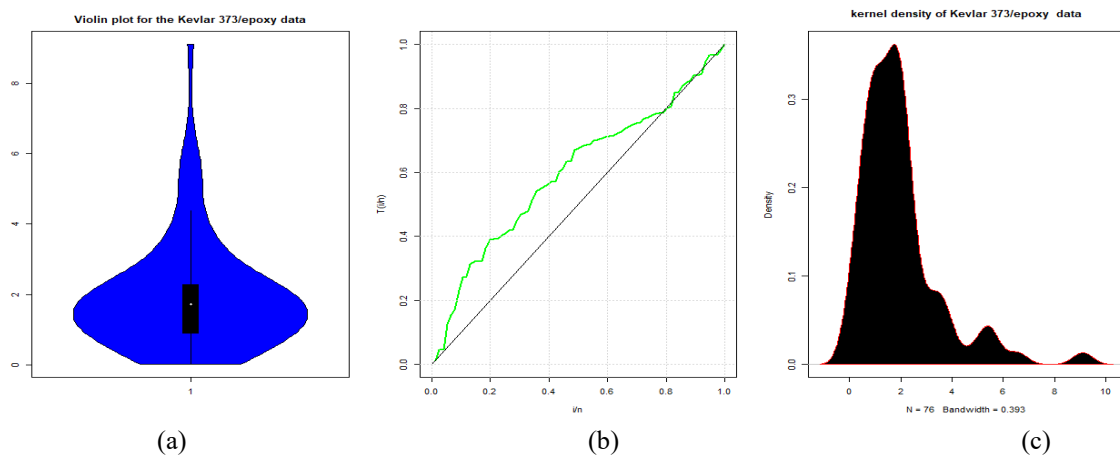


Figure 3. Violin, TTT and Kernel density plot for Kevlar epoxy data

Table 1.0 Maximum likelihood estimate for the Log-LIE model

Model	ζ	ϑ	δ	θ
EPWM	15.3679(1.7301)	3.9213(0.2958)	14.8563(4.2091)	0.3480(0.0453)
EPD	3.5563(0.2945)	0.8667(0.0746)		

Table 2. Measures of the goodness of fit for the Kevlar 373/epoxy data

Model	$-2l$	AIC	BIC	CAIC	HQIC	K	W	P
WEPD	122.738	253.476	262.799	254.040	257.202	0.1071	0.1383	0.3245
EPD	127.493	258.985	263.647	259.150	260.848	0.1533	0.2513	0.0502

Since the WEPD model contains the smallest AIC, BIC, CAIC, HQIC, K, W and P values is considered a better model than the EPD.

V. Conclusion

In this study, a three-parameter model called Weibull Exponential Power distribution is proposed and studied. Various properties of WEPD model are derived. Maximum likelihood estimation method is used to estimate the parameters of the model. The application of WEPD model is demonstrated by using Kevlar 373/epoxy data and it provides a reasonably good parametric fit to the data set than its sub-model because the WEPD model contains the minimum information loss as a result of its smallest AIC, BIC CAIC, K HQIC and P values.

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