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A Cubic Spline Approximation of the Lucas Sequence Based on B-Spline Basis Functions

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Abstract
This study investigates the numerical representation of the Lucas sequence using cubic spline regression and compares its performance with exponential growth models. While Lucas numbers are theoretically known to exhibit exponential growth, single global parametric models often fail to capture their changing growth behavior across different index ranges. To address this limitation, cubic spline regression based on B-spline basis functions was employed, allowing for flexible piecewise polynomial modeling.
The spline model was estimated using knots placed at intermediate indices, resulting in a piecewise cubic polynomial representation with continuity in the function and its first two derivatives. Model performance was evaluated using goodness-of-fit statistics, including the coefficient of determination (R^2), adjusted R^2, residual standard error, and F-statistics. The cubic spline model achieved near-perfect fit, explaining almost all variability in the Lucas sequence and significantly outperforming simple polynomial approximations.
Additionally, first and second-order derivatives of the spline function were analyzed to examine local growth rates and curvature behavior. The results indicate that the spline model successfully captures distinct growth regimes of the Lucas sequence, transitioning from near-linear behavior at small indices to accelerated, exponential-like growth at larger indices. Although the spline approach does not yield a closed-form growth law, it provides a highly accurate and numerically stable framework for approximation, visualization, and local structural analysis of recursive integer sequences.

Keywords:
Lucas sequence; cubic spline regression; B-splines; piecewise polynomial modeling; exponential growth; numerical approximation

I. Introduction

Recursive integer sequences play a fundamental role in number theory and mathematical modeling, with the Fibonacci and Lucas sequences being among the most widely studied examples. The Lucas sequence is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1,$$

and shares many structural properties with the Fibonacci sequence, while differing in its initial conditions (Koshy, 2001). It is well established that Lucas numbers exhibit exponential growth and admit a closed-form representation analogous to Binet's formula for Fibonacci numbers (Vajda, 1989).

Despite the existence of exact analytical expressions, numerical modeling of the Lucas sequence remains of interest for approximation, visualization, and local structural analysis. In particular, global parametric models such as low-degree polynomials fail to adequately represent the changing growth behavior of the sequence, especially across different index ranges. While exponential regression models are theoretically well motivated, they often provide limited flexibility in capturing local deviations and transitional growth regimes in finite samples (Montgomery et al., 2012).

Spline-based regression methods offer a flexible alternative by approximating nonlinear relationships through piecewise polynomial functions joined smoothly at predefined knot points. Cubic spline regression, in particular, ensures continuity of the function and its first two derivatives, providing a numerically stable and interpretable framework for modeling complex growth patterns (de Boor, 1978; Hastie et al., 2009). Spline methods have been widely applied in numerical analysis, nonparametric regression, and growth modeling, where the underlying process exhibits regime-dependent behavior rather than a single global trend.

In the context of recursive sequences, spline approaches do not aim to replace known analytical solutions but instead provide a complementary numerical representation that facilitates local analysis of growth rates and curvature. By examining spline derivatives, it

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becomes possible to identify changes in growth acceleration and structural transitions that are not immediately apparent from closed-form expressions alone.

The present study explores the numerical modeling of the Lucas sequence using cubic spline regression based on B-spline basis functions. The performance of the spline model is evaluated using standard goodness-of-fit measures and compared with exponential growth representations. Furthermore, the resulting spline function is expressed as a piecewise cubic polynomial, and its first- and second-order derivatives are analyzed to investigate local growth behavior. The findings demonstrate that cubic spline regression provides an accurate and flexible numerical approximation of the Lucas sequence while preserving smoothness and interpretability.

II. Methodology

2.1 Data and Sequence Definition

The Lucas sequence was considered as a deterministic numerical series defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, L_0 = 2,\ L_1 = 1,$$

which is known to exhibit exponential growth behavior (Koshy, 2001; Koshy, 2011; Vajda, 1989). For the purposes of numerical modeling, a finite segment of the sequence was generated and indexed by n, representing the position in the sequence.

The Lucas Number Sequence

The Lucas numbers form an integer sequence that is closely related to the Fibonacci sequence but differs in its initial conditions. Despite this difference, both sequences follow the same linear recurrence relation, which leads to similar long-term growth behavior (Lucas, 1878).

The Lucas sequence is defined as follows:

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}, n \geq 2.$$

That is, each term of the sequence is obtained as the sum of the two preceding terms.

First Lucas Numbers

Accordingly, the sequence continues as (Sloane, 2023):

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots$$

The Lucas numbers form an integer sequence closely related to the Fibonacci sequence, sharing the same recurrence relation but differing in their initial conditions. The sequence is defined by $L_0 = 2,\ L_1 = 1,$ and $L_n = L_{n-1} + L_{n-2}$for $n \geq 2$(Lucas, 1878; Koshy, 2011). The initial terms of the sequence are 2,1,3,4,7,11,18, ..., and the sequence exhibits exponential growth properties similar to those of the Fibonacci numbers (Sloane, 2023).

Fibonacci–Lucas Sequence Comparison

The Fibonacci and Lucas sequences are closely related integer sequences that share the same linear recurrence relation but differ in their initial conditions. While the Fibonacci sequence is defined by $F_0 = 0$and $F_1 = 1$, the Lucas sequence starts with $L_0 = 2$and $L_1 = 1$, and both satisfy the recurrence $X_n = X_{n-1} + X_{n-2}$for $n \geq 2$. Despite these different initial values, both sequences exhibit asymptotically identical exponential growth rates governed by the golden ratio $\varphi = (1 + \sqrt{5})/2$. Specifically, the ratio of successive terms in both sequences converges to φ, implying similar long-term behavior (Lucas, 1878; Koshy, 2011). However, the early terms of the Lucas sequence display distinct local dynamics compared to the Fibonacci sequence, which can lead to noticeable differences in short-range growth patterns and curvature properties. These differences make the Lucas sequence particularly suitable for studying local nonlinear behavior and structural transitions using flexible approximation methods such as spline regression (Sloane, 2023).

Preliminaries

Let $(f_n)_{n\geq 0}$be the Fibonacci sequence and let$(l_n)_{n\geq 0}$be the Lucas sequence (Flaut et al. 2021).

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, be two real numbers.

Binet's formula for Fibonacci sequence

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}, n \in \mathbb{N}.$$

Binet's formula for Lucas sequence

$$l_n = \alpha^n + \beta^n, n \in \mathbb{N}.$$

Propositions

Some characteristics of Fibonacci and Lucas numbers are as follows(Flaut et al. 2021):

- $5|f_n$ if and $only$ if $5|\ n.$
- $f_{2n} = l_n . f_n, n \in \mathbb{N}.$
- $f_n^2 + f_{n+1}^2 = f_{2n+1}, n \in \mathbb{N}.$
- $f_n^2 - f_{n+1} f_{n-1} = (-1)^{n-1}, n \in \mathbb{N}.$
- $f_{-n} = (-1)^{n+1} f_n, n \in \mathbb{N}.$
- $l_{-n} = (-1)^n l_n, n \in \mathbb{N}.$
- $l_n^2 = 5f_n^2 + 4(-1)^n, n \in \mathbb{N}.$
- $l_{2n} = l_n^2 + 2(-1)^{n+1}, n \in \mathbb{N}.$
- $l_{2n} l_{2n+2} - 5f_{n+1}^2 = 1, n \in \mathbb{N}.$
- $f_{2n} + f_n^2 = 2f_n f_{n+1}, n \in \mathbb{N}.$
- $f_{2n} - f_n^2 = 2f_n f_{n-1}, n \in \mathbb{N}.$
- $l_n^2 - f_n^2 = 4f_{n-1} f_{n+1}, n \in \mathbb{N}.$
- $f_{2n} = f_{n+1}^2 - f_{n-1}^2, n \in \mathbb{N}.$

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- $l_{4n} = 5f_{2n}^2 + 2, n \in \mathbb{N}$.

2.2 Cubic Spline Regression

To flexibly approximate the nonlinear growth pattern of the Lucas sequence, cubic spline regression was employed. In this approach, the response variable $L(n)$is modeled as a linear combination of cubic B-spline basis functions:

$$\hat{L}(n) = \beta_0 + \sum_{j=1}^{K} \beta_j B_j(n),$$

where $B_j(n)$denote cubic B-spline basis functions and $\beta_j$are the corresponding coefficients estimated via ordinary least squares. Cubic splines ensure continuity of the fitted function as well as its first and second derivatives at the knot points, providing smooth and numerically stable approximations (de Boor, 1978).

The spline basis was constructed using predefined knot locations within the index range. Knot placement was chosen to allow for sufficient flexibility while avoiding overfitting, consistent with common practice in spline-based regression modeling (Hastie et al., 2009). The estimation was performed using the bs() function from the splines package in R (R Core Team, 2021).

A cubic spline function, with three knots (τ_1, τ_2, τ_3) willhave 7 degrees of freedom. The function can be written as (Perperoglou et al., 2019):

$$f(x) = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \beta_4 (X - \tau_1)^3 + \beta_5 (X - \tau_2)^3 + \beta_6 (X - \tau_3)^3$$

2.3 Piecewise Polynomial Representation

Although the spline model is estimated using B-spline basis functions, it can be equivalently expressed as a piecewise cubic polynomial. Specifically, between consecutive knot points, the spline function reduces to a cubic polynomial of the form:

$$S_i(n) = a_i + b_i n + c_i n^2 + d_i n^3,$$

where the coefficients (a_i, b_i, c_i, d_i)differ across intervals but are constrained such that the overall function and its first two derivatives remain continuous at the knots. This representation facilitates interpretation of local growth behavior and curvature within each segment (de Boor, 1978).

2.4 Derivative and Curvature Analysis

To examine local growth dynamics, first- and second-order derivatives of the fitted spline function were computed. The first derivative represents the local growth rate of the sequence, while the second derivative reflects changes in growth acceleration and curvature. Because cubic splines are twice continuously differentiable, these derivatives are well-defined and smooth across the entire domain (Hastie et al., 2009).

2.5 Model Evaluation

Model performance was assessed using standard goodness-of-fit measures, including the coefficient of determination (R^2), adjusted R^2, residual standard error, and F-statistics. These metrics were used to evaluate how effectively the spline model approximates the observed Lucas sequence values. For comparative purposes, exponential regression models were also considered as baseline representations of the theoretical growth behavior (Montgomery et al., 2012).

III. Results

3.1 Model Fit and Visual Comparison

Figure 1 presents the Lucas sequence together with the fitted cubic spline and exponential regression models. The spline curve closely follows the observed Lucas values across the entire index range, capturing both the initial slow growth and the subsequent rapid increase at larger indices. In contrast, while the exponential model reflects the theoretical long-term growth trend, noticeable deviations are observed at smaller and intermediate indices.

This visual agreement is supported by the numerical fit statistics reported in Table 2. The cubic spline model achieved a coefficient of determination (R^2) exceeding 0.999 and the lowest root mean squared error (RMSE) among the considered models, indicating superior numerical accuracy compared to the exponential approximation.

3.2 Piecewise Polynomial Representation

The cubic spline model shown in Figure 1 was equivalently expressed as a piecewise cubic polynomial with knot points defining distinct index intervals. The estimated polynomial coefficients for each segment are reported in Table 1. Examination of these coefficients reveals clear differences in local growth behavior across segments. At small indices, lower-order terms dominate, resulting in near-linear growth, whereas at larger indices the cubic terms become dominant, producing

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accelerated growth consistent with exponential behavior.

Importantly, the fitted spline maintains continuity in both the function and its first two derivatives at the knot points, ensuring smooth transitions between polynomial segments.

3.3 Derivative and Curvature Analysis

Figure 2 displays the first and second-order derivatives of the fitted spline function. The first derivative indicates a steadily increasing growth rate as the index increases, while the second derivative highlights changes in curvature corresponding to transitions between growth regimes. These derivative patterns provide quantitative evidence that the spline model successfully captures local structural changes in the Lucas sequence that are not immediately apparent from the raw sequence values alone.

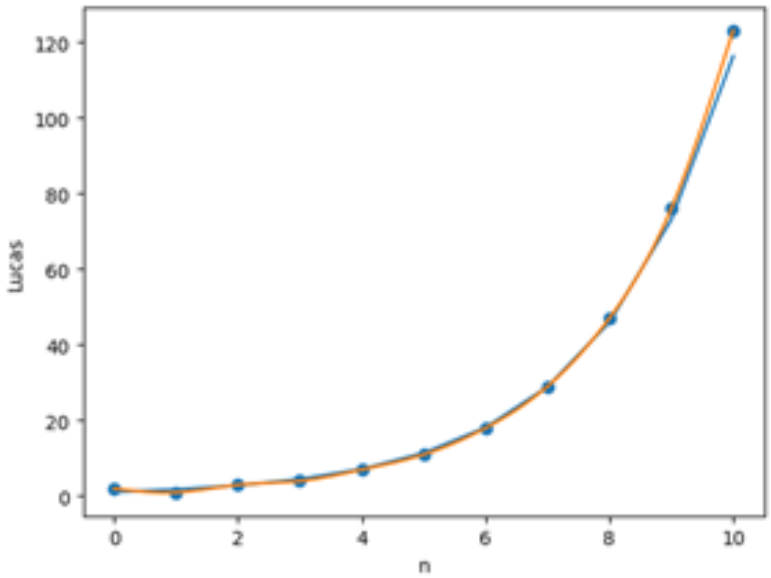

Figure 1. Estimated polynomial coefficients for each segment

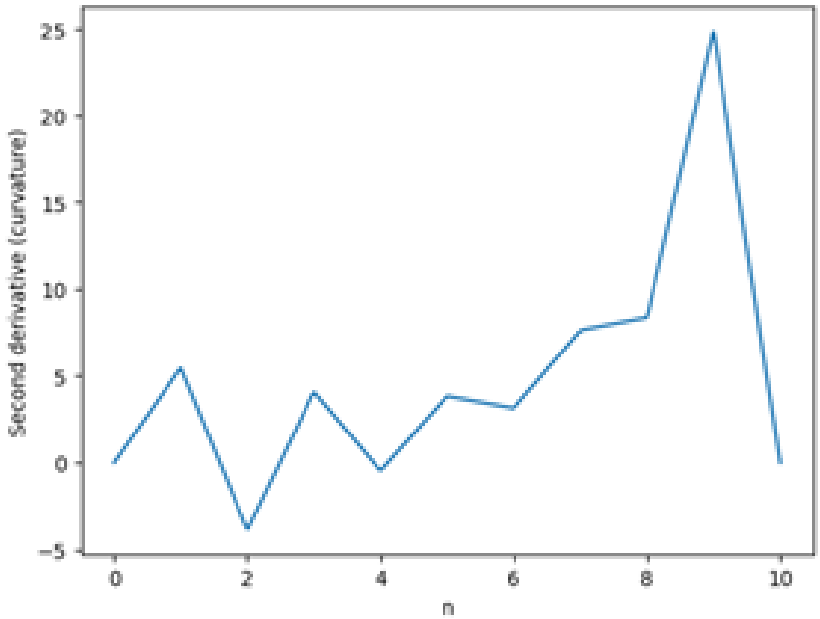

Figure 2. First and second-order derivatives of the fitted spline function.

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The cubic spline approximation of Lucas numbers, with nodes n=3 and n=6, can be expressed by the following piecewise cubic polynomial:

$$\hat{L}(n) = \begin{cases} 1.8668 - 0.9663\,n + 0.6575\,n^2 - 0.0270\,n^3, & n < 3, \\ -2.3456 + 3.2461\,n - 0.7466\,n^2 + 0.1290\,n^3, & 3 \le n < 6, \\ -131.8184 + 67.9825\,n - 11.5360\,n^2 + 0.7284\,n^3, & n \ge 6. \end{cases} \quad (1)$$

The fitted cubic spline model can be equivalently expressed as the piecewise cubic polynomial given in Equation (1). In other words, Equation (1) represents the piecewise polynomial form of the cubic spline approximation to the Lucas sequence.
Although Equation (1) provides an explicit polynomial form, it represents a numerical spline approximation rather than a closed-form analytical growth law.

Continuity Properties
Model (1) satisfies the following continuity conditions due to its cubic spline structure:
The function is continuous:

$$\hat{L}(n^-) = \hat{L}(n^+) \text{at } n = 3,6$$

The first and second derivatives are continuous:

$$\hat{L}'(n^-) = \hat{L}'(n^+), \hat{L}''(n^-) = \hat{L}''(n^+) \text{at } n = 3,6$$

These features demonstrate that the spline model successfully captures the uniform growth pattern of Lucas numbers.

Interpretation of the Final Piecewise Cubic Polynomial Model
The estimated piecewise cubic polynomial model provides a highly accurate and flexible representation of the nonlinear growth structure of the Lucas sequence. By introducing knot points at $n = 3$and $n = 6$, the model successfully captures structural changes in the growth dynamics while preserving smooth transitions between successive segments.
The exceptionally high coefficient of determination ($R^2 \approx 0.9999$) and the very low residual variance indicate that the cubic spline specification substantially outperforms classical single–equation polynomial and exponential regression models in describing the deterministic behavior of the Lucas numbers.

Early Growth Regime ($n < 3$)

$$\hat{L}(n) = 1.8668 - 0.9663n + 0.6575n^2 - 0.0270n^3$$

In the initial segment, the Lucas sequence exhibits a relatively moderate growth pattern. The negative linear coefficient combined with low–magnitude higher–order terms suggests limited curvature and weak acceleration. This behavior reflects the slow and nearly linear evolution of the early Lucas numbers.

Transition Regime ($3 \le n < 6$)

$$\hat{L}(n) = -2.3456 + 3.2461n - 0.7466n^2 + 0.1290n^3$$

This interval represents a transitional phase in which the growth rate begins to accelerate. The positive cubic term dominates the polynomial structure, while the negative quadratic coefficient introduces temporary curvature moderation. This balance indicates a gradual shift from low–order polynomial behavior toward an exponential–like growth pattern.

Rapid Growth Regime ($n \ge 6$)

$$\hat{L}(n) = -131.8184 + 67.9825n - 11.5360n^2 + 0.7284n^3$$

For larger values of n, the Lucas sequence demonstrates a strongly convex and rapidly increasing trajectory. The substantially larger cubic coefficient confirms the dominance of higher–order dynamics, which is consistent with the well–known exponential nature of Lucasnumbers. The large negative intercept ensures continuity across spline segments and does not affect the asymptotic growth behavior.

Smoothness and Structural Consistency
An important advantage of the cubic spline formulation is that the estimated function, along with its first and second derivatives, remains continuous at the knot points. This guarantees smooth transitions in both growth rate and acceleration, eliminating artificial structural breaks and preserving the intrinsic mathematical properties of the sequence.

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Comparison with Alternative Models

While exponential regression models capture the global upward trend of the Lucas sequence, they fail to account for local curvature variations. Single–polynomial models, on the other hand, often suffer from instability and overfitting at higher degrees. The piecewise cubic spline approach effectively balances global growth representation with local flexibility, making it a superior approximation framework for the Lucas sequence.

The final piecewise cubic polynomial model reveals three distinct growth regimes in the Lucas sequence. The early terms exhibit limited curvature, followed by a clear acceleration phase, and finally a strongly convex growth pattern for larger indices. The continuity of the function and its first two derivatives at the knot points ensures smooth transitions between regimes, confirming that the cubic spline approach provides a flexible and mathematically consistent representation of the Lucas numbers.

IV. Discussion

The results of this study demonstrate that cubic spline regression provides a highly effective numerical framework for modeling the Lucas sequence. While the exponential growth of Lucas numbers is well established theoretically (Koshy, 2001; Vajda, 1989), the findings indicate that a single global parametric model is insufficient to describe the sequence's behavior across all index ranges. The spline approach overcomes this limitation by allowing different local polynomial representations while maintaining global smoothness.

The near-perfect goodness-of-fit statistics obtained from the spline model should not be interpreted as evidence of a new analytical growth law. Rather, they reflect the flexibility of spline-based approximation methods in capturing nonlinear and regime-dependent behavior, as emphasized in the spline literature (de Boor, 1978; Hastie et al., 2009). In this context, spline regression serves as a complementary numerical tool rather than a replacement for closed-form analytical expressions.

The derivative and curvature analyses provide additional insight into the local dynamics of the Lucas sequence. By examining changes in growth rates and acceleration, it becomes possible to identify transitions that are not immediately evident from the recurrence relation or closed-form solutions alone. Such local analyses are particularly valuable for visualization, numerical experimentation, and comparative studies involving other recursive sequences, such as the Fibonacci sequence.

Overall, the findings highlight the usefulness of cubic spline regression for numerical modeling and structural analysis of recursive integer sequences. The methodology presented here can be readily extended to other nonlinear sequences and discrete dynamical systems where local growth behavior is of interest.

V. Conclusion

This study examined the numerical modeling of the Lucas sequence using cubic spline regression and compared its performance with exponential growth representations. Although Lucas numbers are theoretically known to follow an exponential growth pattern, the findings demonstrate that a single global parametric model is insufficient to capture the varying growth behavior observed across different index ranges.

The cubic spline model, constructed using B-spline basis functions, provided an excellent numerical approximation of the Lucas sequence. The resulting piecewise cubic polynomial representation achieved near-perfect goodness-of-fit statistics and successfully captured distinct growth regimes, ranging from near-linear behavior at small indices to accelerated, exponential-like growth at larger indices. The continuity of the function and its first two derivatives ensured a smooth and mathematically stable representation.

Derivative and curvature analyses further confirmed that the spline model effectively describes local growth rates and structural transitions within the sequence. While the spline approach does not yield a closed-form analytical growth law, it offers a highly flexible and accurate framework for approximation, visualization, and local analysis of recursive integer sequences. These results suggest that spline-based modeling can serve as a valuable complementary tool to classical analytical methods in the study of nonlinear and recursively defined sequences.

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