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Application of Legendre Quadrature Based Linearization Technique for Solving Electrochemical and Combustible Fluid Models

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ABSTRACT
An iterative linearization method was established utilizing Legendre quadrature within a moment minimization framework, the procedure was executed consecutively to enhance the precision of each iteration. Two modeled equations were solved to examine the approach's performance, and the computed tolerance value indicated that the method is both consistent and accurate. The results of the approach were likewise determined to be favorable when compared to cited literature.

Key words: Iterative linearization; Legendre quadrature, moment method; Tolerance.

I. INTRODUCTION

Gaining a comprehensive understanding of industrial products like thermal energy provides several advantages, such as decreasing our dependence on fossil fuels and so minimizing the damaging effects of carbon emissions, providing heat for houses and manufacturing power via steam generators. Nevertheless, researchers have resorted to numerical and theoretical analysis of systems utilizing thermal energy due to their exorbitant cost and lengthy time required for experimental evaluation to be completed. Examples of such researches can be found in the work of authors in [1-3].It is important to acknowledge that the majority of situations that represent thesephysical systems under consideration are typically converted into differential equations in order to be studied numerically. However, it is necessary to solve these differential equations in order to conduct a more in-depth analysis of the system. Unfortunately, the majority of these equations are nonlinear, which significantly increases the complexity of solving them.This difficulty has prompted the creation of many numerical methods designed for specific classes of differential equations, examples of these techniques are Adomian decomposition method [4-5], homotopy perturbation method [6] which have been applied to problems in fluid mechanics and combustible fluid dynamics. The AkbariGanji method [7] is another numerical procedure that has been employed in the recent time for problems in the area of electrochemistry.

The implementation of various methodology in applied sciences and engineering has provided evidence for the reliance of theoretical engineering applications on numerical methods. This study therefore aims to present an alternative numerical approach to solve models derived from the theoretical analysis of engineering problems. It seeks to enhance the linearization currently used. This linearization considers all components of the model, unlike the conventional linearization as well as giving room to improve on the results by repeating procedures iteratively. It also incorporates a minimization procedure that spans the entire domain by using Legendre quadrature method. This is necessary due to the strong nonlinearity of the problems in applied sciences.

II. PROCEDURE OF LEGENDRE QUADRATURE BASED LINEARIZATION TECHNIQUE

Suppose the general combustible model is represented by the differential equation

$$L\theta(x)+R\theta(x)+N\theta(x)=g(x) \qquad (1)$$

Subject to the boundary condition $B(\mu(x))=c_i,\ i=1,2,..$ where L is the highest order derivative in the differential equation, $\theta(x)$ is the temperature distribution, R is the remaining linear derivative(s), N is the non-linear function including derivatives and $g(x)$ is the source term; The initial stage of the procedure requires linearizing the nonlinear function by making use of the Taylor series expansion for a single variable [8]. That is equation (1) becomes

$$L\theta(x)+R\theta(x)+N\theta(x)\big|_{Linear}-g(x)=0. \quad (2)$$

Equation (2) becomes a linear differential equation which will be solved by assuming a polynomial function;

$$\theta(x)=\sum_{i=0}^{n}c_i x^i \quad (3)$$

Where c_i are constants to be determined. Imposing equation (3) on the boundary conditions, give rise to a system of algebraic equations whose number depends on the boundary conditions. The next step is to substitute the assumed $\theta(x)$ into equation (2), to generate the residual function $E(x)$, with the intention of minimizing $E(x)$ to zero throughout the entire domain. In order to minimize $E(x)$, the method of moment which involves multiplying $E(x)$ by the weight function prior to integrating over the domain is employed. This involves

$$\int_{b_1}^{b_2} E(x).x^i dx = 0, \quad b_1 \le x \le b_2 \;\; i=0,1,2,..n \quad (4)$$

Given the intricate nature of the integration in equation (4), the Gauss-Legendre method is utilized. This approach is a quadrature formula that does not necessitate the integration region to have equal spacing. Noting that this formula is [9,10],

$$\int_{-1}^{1} f(z)dz \approx \sum_{q=1}^{m} B_q f(z_q), \quad (5)$$

where z_q are the roots of the Legendre polynomial defined within $[-1,1]$ and m is the order of the polynomial, in the case of this work, $m=6$ is employed. It is worthy to note that not every problem under consideration will be bounded by $[-1,1]$, as a result integration within $[b_1,b_2]$ requires a transformation of the original Legendre formula. It then implies that equation (4) becomes

$$\int_{b_1}^{b_2} x^i\, E(x)dx = \frac{b_2-b_1}{2}\sum_{q=1}^{m}\frac{\left((b_1+b_2)+(b_2-b_1)z_q\right)^i}{2}E\left(\frac{\left((b_1+b_2)+(b_2-b_1)z_q\right)}{2}\right)A_q \quad (6)$$

Equation (6), along with the algebraic equations derived from the imposition of trial functions involve constants c_i. These constants equations are then subsequently solved to obtain the constants and consequently the initial approximation of the iteration becomes

$$\theta_0=\sum_{i=0}^{n}c_i x^i \quad . \quad (7)$$

The process is thereafter performed for the iterative sequence

$$L\theta_n(x)+R\theta_n(x)+N\theta_{n-1}(x)-g(x)=0, \; n=1,2,3,... \quad (8)$$

Equation (8) guarantees that at each iteration, the differential equation that needs to be solved will be linear. The linearity is ensured by utilizing the previous solution in the nonlinear function for the succeeding iterations. This procedure give $\theta_1,\theta_2,\theta_3,...$ until the desired accuracy is achieved. Due to the fact that this procedure is iterative based, then the method employed in [8] is deployed to assess the accuracy. This approach comprises the computation of the tolerance value χ_n, which is given by

$$\chi_n=\left|\theta_n-\theta_{n-1}\right| \quad (9)$$

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The term χ_n is considered to be very small as it is the criterion for determining when to terminate the iteration.

III. APPLICATIONS TO MODELLED EQUATIONS

Two mathematical equations representing model problem from electrochemistry and combustible fluid dynamics will be employed as numerical examples to illustrate the applicability as well as the efficiency of the approach.

3.1. **Illustration one**: Consider the combustible fluid model, given as [2]

$$1+\frac{d}{dy}\left(e^{-\alpha\theta}\frac{du}{dy}\right)+6\gamma\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}+Gr\left(\theta+\sigma\theta^2\right)=0,\ u(0)=0=u(1), \tag{10}$$

$$\frac{d}{dy}\left(e^{-\eta\theta}\frac{d\theta}{dy}\right)+\lambda\left((1+\varepsilon\theta)^m e^{\frac{\theta}{1+\varepsilon\theta}}+\delta\left(\frac{du}{dy}\right)^2\left(e^{-\alpha\theta}+2\gamma\left(\frac{du}{dy}\right)^2\right)\right)=0, \tag{11}$$

$$\text{with the boundary: } \left(e^{-\eta\theta}\frac{d\theta}{dy}-Bi_1\theta\right)\bigg|_{y=0}=0=\left(e^{-\eta\theta}\frac{d\theta}{dy}+Bi_2\theta\right)\bigg|_{y=1}$$

Linearizing equations (10) and (11), then

$$1+\frac{d}{dy}\left(\frac{du_0}{dy}\right)+6\gamma\left(\frac{du_0}{dy}\right)+Gr\left(\theta_0+\sigma\theta_0\right)=0,\quad u_0(0)=0=u_0(1), \tag{12}$$

$$\frac{d}{dy}\left(\frac{d\theta_0}{dy}\right)+\lambda\left(\left(1+\varepsilon\theta_0\right)+\delta\left(1-\alpha\,\theta_0+2\gamma\left(\frac{du_0}{dy}\right)\right)\right)=0, \tag{13}$$

with the iterative sequence,

$$1+\frac{d}{dy}\left(e^{-\alpha\theta_{n-1}}\frac{du_n}{dy}\right)+6\gamma\left(\frac{du_{n-1}}{dy}\right)^2\frac{d^2u_n}{dy^2}+Gr\left(\theta_{n-1}+\sigma\theta_{n-1}{}^2\right)=0,\quad u_n(0)=0=u_n(1), \tag{14}$$

$$\frac{d}{dy}\left(e^{-\eta\theta_{n-1}}\frac{d\theta_n}{dy}\right)+\lambda\left((1+\varepsilon\theta_{n-1})^m e^{\frac{\theta_{n-1}}{1+\varepsilon\theta_{n-1}}}+\delta\left(\frac{du_{n-1}}{dy}\right)^2\left(e^{-\alpha\theta_{n-1}}+2\gamma\left(\frac{du_{n-1}}{dy}\right)^2\right)\right)=0 \tag{15}$$

$$\chi_n=\left|u_n-u_{n-1}\right|,\quad n=1,2,3,4,5.$$

Employing the procedure discussed in section two to solve equations (12) to (15) with the attached boundary conditions then $u_i,\ \theta_i, i=0,1,2,3,4,5$ are adequately obtained.

3.2. **Illustration two**: Consider a theoretical framework that encompasses the movement of mass in an electrochemical reactor given as [7]:

$$\frac{d^2W(x)}{dx^2}+\frac{s}{x}\frac{dW(x)}{dx}=\phi^2W(x)\exp\left(\frac{\gamma\beta(1-W(x))}{1+\beta(1-W(x))}\right),\ W(1)=1,\quad W'(0)=0. \tag{16}$$

The same approach is employed to obtain $W_0,\ W_1,\ W_2,\ W_3$ for various s,ϕ,γ,β.

$$\chi_n=\left|W_n-W_{n-1}\right|,\quad n=1,2,3$$

IV. DISCUSSION OF RESULTS

The solution of physical models pertaining to electrochemistry and combustible fluid dynamics has been achieved in polynomial form utilizing

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Legendre quadrature and the iterative linearization technique. The tolerance behavior, representing the difference in solutions between succeeding steps, is illustrated in Figures 1 and 2 for u and θ, respectively, in Problem 1.It was noted that these tolerances diminish with each iteration and become negligible by the fifth iteration in the figures. This decline ensures the method's consistency and accuracy, as supported by the characteristics of any iterative approach.This observation was similarly noted in Figure 3.This behavior is attributable to eachnext iteration representing an enhancement over the preceding one. Tables 1 and 2 also show the comparison in the results of the present method with the method in literature, and it was observed that there is agreement between the compared results.

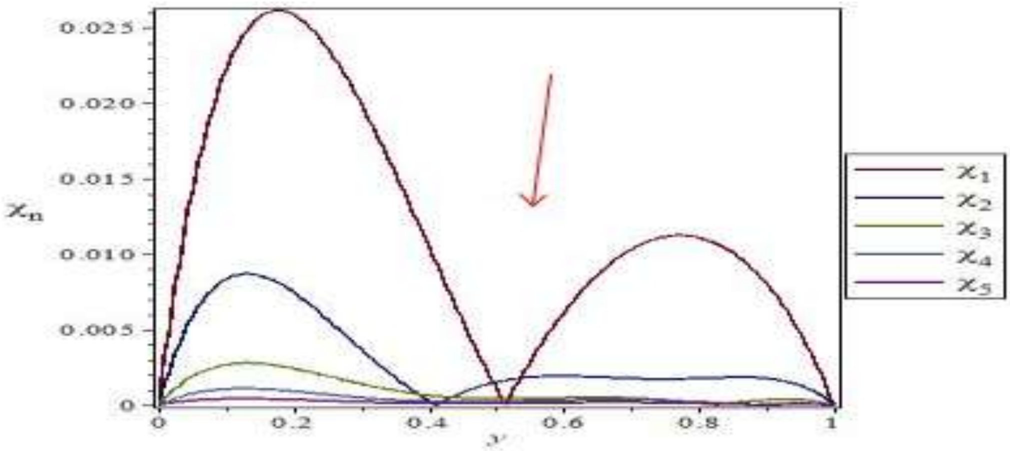

Figure 1: Declining tolerance value for u of each iteration in problem 1

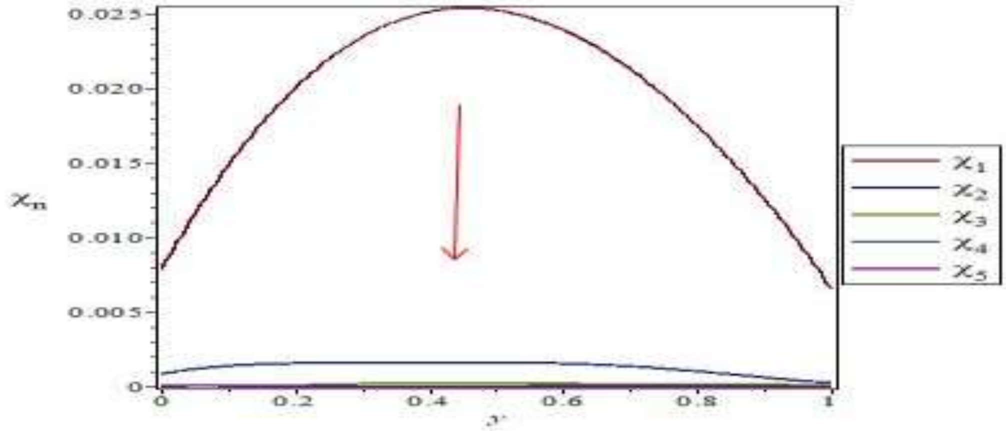

Figure 2: Declining tolerance value for θ of each iteration in problem 1

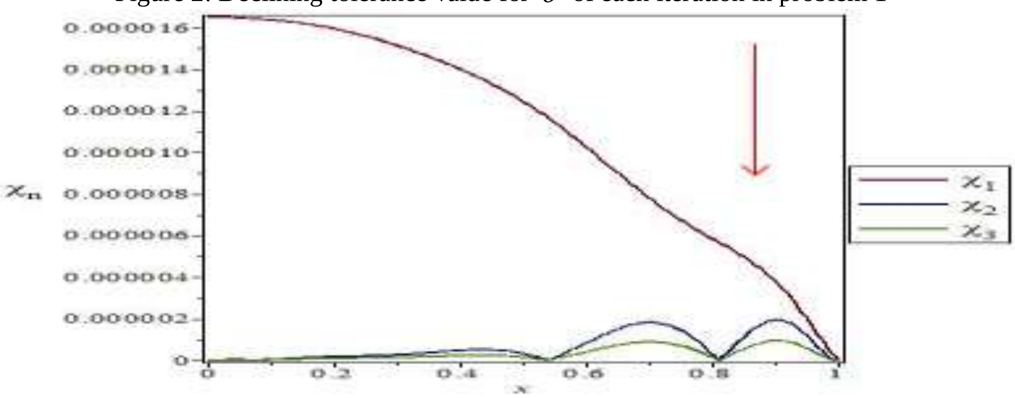

Figure 3: Declining tolerance value for W of each iteration in problem 2

Table 1: Comparison of results for u(y) and $\theta(y)$ with referenced literature

y	u_5	[2]	θ_5	[2]
0.0	0.000000000000	$-6.96195549259\times10^{-18}$	0.0221170432015	0.0221518490244846
0.2	0.078062473074	0.0781363407729733	0.0573406433703	0.0575154443477470
0.4	0.120542486668	0.1200699036038268	0.0747933091330	0.0750593058822556
0.6	0.120502186275	0.1200699036038268	0.0748034877050	0.0750593058822556
0.8	0.078596839606	0.0781363407729733	0.0573332842621	0.0575154443477470
1.0	0.000000000000	$-7.69196285588\times10^{-18}$	0.0220879224633	0.0221518490244846

Table 2: Comparison of results for W(x) with referenced literature

	$\phi=0.1$		$\phi=5$	
x	w_3	[7]	w_3	[7]
0.0	0.9983187294	0.9983	0.0663705369	0.0625
0.2	0.9983859341	0.9984	0.0781595635	0.0774
0.4	0.9985876405	0.9986	0.1208412890	0.1290
0.6	0.9989237825	0.9989	0.2236610542	0.2246
0.8	0.9993945307	0.9994	0.4586489011	0.4795
1.0	0.9999999975	1.0000	1.0000000000	1.0000

V. CONCLUSION

An unevenly spaced quadrature, specifically Legendre, has been employed to develop an iterative linearization method for addressing nonlinear problems in applied sciences. The method displays results in polynomial form, so eliminating the influence of discretization error. Two problems from literature were addressed using the developed method to assess its applicability and efficacy. The findings indicated that the approach is reliable and accurate.This method is thus recommended for addressing nonlinear mathematical models originating from engineering and sciences, specifically in the context of boundary value problems.

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