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Consistency, Stability, and Convergence Analysis of a Finite Difference–Based Method of Lines with Five-Point Discretization

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ABSTRACT
This paper presents a detailed investigation into the consistency, stability, and convergence properties of the finite difference–based Method of Lines (FD-MoL) employing a five-point discretization for the numerical solution of parabolic partial differential equations. The heat equation without a source term is adopted as a benchmark problem to illustrate the methodology and theoretical analyses. In the FD-MoL framework, spatial derivatives are discretized using a high-order central finite difference scheme, while the temporal domain remains continuous, allowing for the application of robust ordinary differential equation (ODE) solvers.The consistency of the proposed scheme is established through local truncation error analysis, revealing first-order accuracy in time and second-order accuracy in space. Numerical stability is examined using Von Neumann Fourier analysis, with results showing that the implicit formulation is unconditionally stable for all wave numbers and time-step sizes. Convergence is verified both analytically and in accordance with the Lax equivalence theorem, confirming that the numerical solution converges to the exact solution as the discretization parameters approach zero.The study highlights the effectiveness of the five-point stencil in enhancing spatial resolution while maintaining numerical robustness. These properties make the scheme highly suitable for large-scale and long-time simulations in science and engineering domains where high spatial accuracy and unconditional stability are essential.

KEYWORDS: Method of Lines, finite difference method, five-point discretization, stability, consistency, convergence, heat equation, Von Neumann analysis

1. INTRODUCTION

Partial differential equation (PDE)is one of the tools to demonstrate how some quantities vary with position and time[1]. One of such equations is the heat equation which is used to describe the variation of temperature in abody.Several approximate methods have been investigated to solve time dependent partial differential equation [2]. The complexity of many real situations makes it difficult to obtain analytical solutions, hence, numerical methods are being deployed. [1], [3],[4],[5], [6]. The main subject in numerical analysis is to study these methods in terms of their stability, consistency, convergence and order of accuracy.

One of the techniques to solve a time-dependent Partial Differential Equations is the application of Method of Lines (MoL). Method of Lines has formed a broad interest in Science and Engineering. It discretizes the spatial dimension by using techniques such as Finite difference, Finite element, Finite volume, Spectral or Meshless methods. It serves as a general procedure for the solution of Partial Differential Equations (PDEs) [7]. The use of MoL yields a system of first order differential equations with initial value[2].This method could be described as a semi analytical procedure and a general way of viewing a Partial Differential Equation as a system of Ordinary Differential Equations (ODEs). [8]. In this paper, we will focus on the finite difference scheme (FDM).

Application of MoL cuts across various problems such as parabolic PDEs [9], reaction diffusion equation [10], Biomedical Science and Engineering, [11], extended Boussinesesq equation [12], one dimensional wave equation subject to an integral conservation condition[13], the conservation laws problem[14], and among others. The basic idea of MoL is to convert the Partial Differential Equations to a system of Ordinary Differential Equations (ODEs) or algebraic equations according to the type of boundary conditions.

Numerical methods compute approximate solutions for differential (DEs). In order for the numerical solution to be a reliable approximation of the given problem, the numerical method should satisfy certain properties. In this paper, we consider properties of numerical methods that are most common in numerical analysis such as consistency, stability and convergence of finite difference-based method of lines (FD-MoL). The basic idea of Finite Difference (FD) schemes is to replace derivatives by FD approximations, however, FD-MoLinvolves discretizing the spatial derivatives while keeping time as a continuous variable, effectively converting PDEs into a system of ODEs. This allows the use of robust time-stepping algorithms, enabling flexibility in time-step selection and ensuring stability across various spatial discretizations.[10].

The stability of the Method of Lines involves examining the behaviour of numerical solutions when applied to partial differential equations.Ensuring numerical stability is crucial for obtaining meaningful results from numerical methods, as it prevents the solution from diverging or exhibiting uncontrolled oscillations. Stability analysis typically evaluates how the selection of spatial discretization techniques such as finite differences and temporal integration methods influence the robustness of the numerical scheme. In particular, the stability of the spatial discretization plays a vital role in maintaining the reliability and accuracy of the numerical solution. This stability is often associated with the eigenvalues of the resulting matrix[15]. The significance of numerical stability cannot be overstated, as unstable methods may lead to solutions that deviate significantly from the exact solution, ultimately resulting in incorrect conclusions.

To solve the system of ordinary differential equations (ODEs) arising from the method of lines (MoL), time integration methods such as Runge-Kutta or multistep schemes are commonly employed. In the case of linear partial differential equations (PDEs), Fourier analysis is a useful tool for assessing the stability of numerical methods. A specialized form of this approach, known as Von Neumann analysis, is used to evaluate the amplification of errors introduced during the discretization process. It is also applicable for analyzing the stability of problems with periodic boundary conditions[16]. The heat equation (or diffusion equation) is frequently employed as a benchmark problem in MoL stability studies. Numerous studies have extensively examined the consistency, stability, and convergence of finite difference schemes, including the works of [17],[18], [19], [15], [20], [16],[21] , [22], among many others. However, the consistency, stability, and convergence of the finite difference-based method of lines have not been thoroughly investigated with respect to five-point discretization.

II. METHODS

To evaluate the numerical solution of nonlinear equations using the FD-MoL, the following steps are considered;

1. Discretize the spatial derivatives in the PDE
2. Formulate the approximate system of ODEs
3. Apply any integration algorithm for initial value of ODE to compute an approximate numerical solution to the PDE

Consider the heat equation without heat source which models the temperature distribution in a long thin rod of length L which is insulated along its length so that there is no heat loss. The partial differential equation (PDE) for this problem with the initial condition is

$$U_t = U_{xx}, \qquad x \in (x_0, L], \; t > 0,$$
$$U(x,0) = f(x), \qquad x \in (x_0, L]. \qquad (1)$$

Evaluating the diffusion operator U_{xx} using a five-point central finite difference, gives

$12h^2U_i' = -U_{i-2} + 16U_{i-1} - 30_i + 16_{i+1} - U_{i+2} + O(h^4), i = 1,2,\ldots, M-1 (2)$

With the initial condition:

$$U_i(0) = f(x_i)$$

Where $h = {(L - x_0)}/{M}$

Equation (4) could be written in the form;

$12h^2U' = AU, \qquad U(0)= [f(0), f(x_i), \ldots, f(x_M)]^T, \qquad (3)$

Where A, for periodic boundary condition is

$U(x_0, t) = U(L, t)$ and $U_x(x_0, t) = U_x(L, t)$

or

$U_0 = U_M, \qquad U_{x|0} = U_{x|M},$

From (4), for $i = M + 2$, we can write

$12h^2U_{M+2}' = -U_M + 16U_{M+1} - 30U_{M+2} + 16U_{M+3} - U_{M+4} \qquad (4)$

Also;

Take $i = 0$

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$12h^2U_0' = -U_{-2} + 16U_{-1} - 30U_0 + 16U_1 - U_2 + O(h^4)$

Take $U_{-2} = U_0 - hU_{x|0} + O(h^4)$

$12h^2U_0' = 16U_{-1} - 31U_0 + 16U_1 - U_2 - hU_{x|0} + O(h^4)$

For $U_{x|0} = U_{x|M}$,

we have;

$12h^2U_0' = 16U_{-1} - 31U_0 + 16U_1 - U_2 - hU_{x|M} + O(h^4)$

Let $U_{x|M} = {(U_M - U_{M-1})}/{h} + O(h)$

$12h^2U_0' = 16U_{-1} - 31U_0 + 16U_1 - U_2 + (U_M - U_{M-1}) + O(h^4)$

But $U_0 = U_M$, hence we have

$$12h^2U_0' = 16U_{-1} - 30U_0 + 16U_1 - U_2 - U_{M-1} + O(h^4) \quad (5)$$

Now, matrix A for the periodic condition according to equations (4) and (5), is obtained as:

A=

$$\begin{pmatrix} -5/2 & 4/3 & -1/12 & 0 & 0 \\ 4/3 & -5/2 & 4/3 & -1/12 & 0 \\ -1/12 & 4/3 & -5/2 & 4/3 & -1/12 \\ 0 & -1/12 & 4/3 & -5/2 & 4/3 \\ 0 & 0 & -1/12 & 4/3 & -5/2 \end{pmatrix}_{MXM} \quad (6)$$

Direct Fd-Mol For Solving Heat Equation Usingthe Five- Point Discretization

Consider the three-dimensional heat equation satisfying the initial condition

$$U_t = U_{xx} + U_{yy} + U_{zz}, \quad x \in (x_0, L_x], \quad y \in (y_0, L_y],\ z \in (z_0, L_z],\ t > 0, \quad (7)$$

$U(x, y, z, 0) = f(x, y, z), \quad x \in (x_0, L_x], \quad y \in (y_0, L_y], z \in (z_0, L_z]$

Evaluating the diffusion operators U_{xx}, U_{yy} and U_{zz} using five point finite difference, gives

$U'_{i,j,k} = \frac{1}{12h^2}(-U_{i-2,j,k} + 16U_{i-1,j,k} + 16U_{i+1,j,k} - Ui+2,j,k$ + $\frac{1}{12h^2}(-U_{i,j-2,k} + 16U_{i,j-1,k} + 16U_{i,j+1,k} -$ $Ui,j+2,k+112h2-Ui,j,k-2+16Ui,j,k-1+16Ui,j,k+1-Ui,j,k+2$ - $30112hx+$ $112hy+$ $112hzUi,j,k$,

$$U_{i,j,k}(0) = f_{i,j,k} = f(x_i, y_j, z_k) \quad (8)$$

Where $h_x = (L_x - x_0)/M_x$, $h_y = (L_y - y_0)/M_y$, $h_z = (L_z - z_0)/M_z$. This system of ODE has solution in the form

$$U(t) = e^{A_3 t}U(0) \quad (9)$$

Where, U(0) and A_3 will be obtained for some boundary conditions.

III. CONSISTENCY, STABILITY AND CONVERGENCE ANALYSIS OF FD-MOL

Consistency

Theorem 1:

A numerical method is considered consistent if the local truncation error (LTE) vanishes as the grid spacing and time step approach zero. [17]Lax and Ritchman(1956), [23]LeVeque (2007), [24]Burden (2010), [25]Morton and Mayer(2005)

Specifically, if

$L(u(x,t)) = 0$represents the PDE and $L_h(u_h)$ is its numerical approximation, then

$L_h(u_h) \to L(u)$ as $h \to 0$, where h denotes the step size

As the step sizes (spatial or temporal) of a numerical method approach zero, the approximation provided by the numerical scheme converges to the original PDE. In other words, a consistent method will produce discrete equations whose behaviour resembles the continuous PDE as the grid spacing $\Delta x, \Delta y,$ or the time step Δt becomes infinitesimally small

Definition

Given the continuous PDE $\mathcal{L}(u(x,t)) = 0$, whrer $\mathcal{L}$ represents the differential operator, then a numerical method that approximates this PDE should reduce to $\mathcal{L}(u(\Delta x, \Delta t)) = 0$ as $\Delta x, \Delta t \to 0$

Theorem 2

For a finite difference method applied to a PDE to be consistent, the local truncation error defined as $LTE = \frac{U(t+k,x)-U(t,x)}{k}$must approach zero as $k, h \to 0$[25].

Consider an equation of the form

$$u_t = au_{xx} \quad (10)$$

By discretization, *where k is the time step size and h is the space step size*, we have;

$$\frac{U_j^{n+1} - U_j^n}{k} - \frac{a(-U_{j-2}^n + U_{j-1}^n - 30U_j^n + 16U_{j+1}^n - U_{j+2}^n)}{12h^2} = 0$$

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The truncation error (T.E) is given by

$$T.E = \frac{U_j^{n+1}-U_j^n}{k} - \frac{a\left(-U_{j-2}^n+U_{j-1}^n-30U_j^n+16U_{j+1}^n-U_{j+2}^n\right)}{12h^2}$$

$$LTE = \frac{U(t+k,x)-U(t,x)}{k} - a\frac{(-u(t,x-2h)+16u(t,x-h)-30u(t,x)+16u(t,x+h)-u(t,x+2h))}{12h^2} \quad (11)$$

By Taylor Series expansion, we have;

$U(t+k,x) = U + kU + {k^2}/{2!}\,U_{tt} + \cdots$

$U(t,x+h) = U + hU_x + {h^2}/{2!}\,U_{xx} \ldots + O(h^3)$

$U(t,x-h) = U - hU_x + {h^2}/{2!}\,U_{xx} \ldots + O(h^3)$

$U(t,x+2h) = U + 2hU_x + h^2U_{xx} \ldots + O(h^3)$

$U(t,x-2h) = U - 2hU_x + h^2U_{xx} \ldots + O(h^3)$

Thus the term $-U(t,x-2h)$ becomes

$-U(t,x-2h) = -U + 2hU_x - h^2U_{xx} \ldots - O(h^3)$

Substituting into LTE(rename), we have

$-\frac{a}{12h^2}\left[(-U-2hU_x+2h^2U_{xx}) + 16\left(U - hU_x + h22\,Uxx - 30\,U + 16\,U + hUx + h22\,Uxx - U + 2hUx + 2h2\,Uxx + Oh3\right.\right.$

$= U_t + kU_t\frac{k^2}{2!}U_{tt} + O(k^2)$ (consistency condition) (12)

$\therefore T.E = O(k, h^2)$

The scheme is first order in time and second order in space

The scheme is more restricted because time step is very small, thus, explicit method is not recommended

Stability Analysis

Theorem 3 (Von Neumann stability theorem)

For a numerical scheme used to solve PDEs, the method is unconditionally stable if the amplification factor $|G(\zeta)|$ associated with the scheme satisfies:

$|G(\zeta)| \leq 1$for all wave numbers ζ

Where $G(\zeta)$ is the factor by which the amplitude of each Fourier mode is multiplied as the solution progresses in time ([24], [25]

A scheme is said to be unconditionally stable if the stability condition $|G(\zeta)| \leq 1$ holds independently for step size h. ([27] ,[23]

Consider the equation of the form

$$u_t = u_{xx} \qquad x \in \mathcal{R}, t > 0 \quad (13)$$

Implicit difference formula can be written as

$$U_j^n = U_j^{n+1} - \lambda\left(-U_{j-2}^{n+1} + 16U_{j-1}^{n+1} - 30U_j^{n+1} + 16U_{j+1}^{n+1} - U_{j+2}^{n+1}\right) \quad (14)$$

[Taking $\lambda = {k}/{12h^2}$]

$$U_j^n = U_j^{n+1} + \lambda U_{j-2}^{n+1} - 16\lambda U_{j-1}^{n+1} + 30\lambda U_j^{n+1} - 16U\lambda_{j+1}^{n+1} + \lambda U_{j+2}^{n+1} \quad (15)$$

$$= (1+30\lambda)U_j^{n+1} + \lambda U_{j-2}^{n+1} - 16\lambda U_{j-1}^{n+1} - 16U\lambda_{j+1}^{n+1} + \lambda U_{j+2}^{n+1} \quad (16)$$

To analyse the stability, we use the Fourier method, where:

$$U_{j+k}^{n+1} = e^{ik\zeta}\,\widehat{U}^{n+1}(\varepsilon) \quad (17)$$

Applying (17) for different shifts, we get;

$U_{j-1}^{n+1} = e^{-i\zeta}\,\widehat{U}^{n+1}(\varepsilon)$

$U_{j+1}^{n+1} = e^{i\zeta}\,\widehat{U}^{n+1}(\varepsilon)$

$U_j^{n+1} = \widehat{U}^{n+1}(\varepsilon)$

$U_{j+1}^{n} = e^{i\zeta}\,\widehat{U}^{n}(\varepsilon)$

$U_j^{n} = \widehat{U}^{n}(\varepsilon)$

$U_j^{n-1} = \widehat{U}^{n-1}(\varepsilon)$

Substituting the above transformations into the finite difference equation (16), we have

$$=(1+30\lambda)\widehat{U}^{n+1} + \lambda e^{-2i\mathcal{E}}\widehat{U}^{n+1} - 16\lambda e^{-i\mathcal{E}}\widehat{U}^{n+1} - 16\lambda e^{i\mathcal{E}}\widehat{U}^{n+1} + \lambda e^{2i\mathcal{E}}\widehat{U}^{n+1} \quad (18)$$

$$U_j^n = (1+30\lambda)\widehat{U}^{n+1} + \lambda(e^{-2i\mathcal{E}} + e^{2i\mathcal{E}})\widehat{U}^{n+1} - 16\lambda(e^{-i\mathcal{E}} + e^{i\mathcal{E}})\widehat{U}^{n+1} \quad (19)$$

Using the Euler's formula, we obtain,

$$U_j^n = (1+30\lambda)\widehat{U}^{n+1} + 2\lambda\cos(2\zeta)\,\widehat{U}^{n+1} - 32\lambda\cos(\zeta)\,\widehat{U}^{n+1} \quad (20)$$

$$=(1+30\lambda)\widehat{U}^{n+1} + 2\lambda\left(1 - \sin^2(\zeta)\widehat{U}^{n+1}\right) - 32\lambda\cos(\zeta)\,\widehat{U}^{n+1}$$

Using Trigonometric identities, we have

$$=(1+30\lambda) + 2\lambda\left(1-\sin^2(\zeta)\right) - 32\lambda\left(1 - sin2\zeta2\right. \quad (21)$$

$$=(1+30\lambda) + 2\lambda - 2\lambda\sin^2(\zeta) - 32\lambda + 64\lambda\sin^2\left({\zeta}/{2}\right)$$

$$=1 + 32\lambda - 2\lambda\sin^2(\zeta) - 32\lambda + 64\lambda\sin^2\left({\zeta}/{2}\right)$$

$$=1 + 64\lambda\sin^2\left({\zeta}/{2}\right)32\lambda - 2\lambda\sin^2(\zeta) \quad (22)$$

For the scheme to be considered stable, the amplification factor G must satisfy

$$|G(\zeta)| \leq 1$$

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This requirement ensures that the numerical solution does not grow unbounded as $n \to \infty$

Given the amplification factor,

$$G = \frac{1}{1+64\lambda sin^2(\zeta/2)32\lambda - 2\lambda sin^2(\zeta)} \widehat{U}^n \quad (23)$$

By mathematical induction;

$$\widehat{U}^n = \frac{1}{1+64\lambda sin^2(\zeta/2)32\lambda - 2\lambda sin^2(\zeta)} \widehat{U}^0 \quad (24)$$

Hence, the amplification factor satisfies

$$\frac{1}{1+64\lambda sin^2(\zeta/2)32\lambda - 2\lambda sin^2(\zeta)} \leq 1 \quad (25)$$

This stability condition aligns with the Von Neumann stability analysis, which states that for an implicit scheme for parabolic PDEs, the amplification factor should be bounded by 1 for all ζ.

Since $G \leq 1 \; \forall \; \zeta$,

then, the implicit finite difference scheme is unconditionally stable

Convergence

Anumerical scheme is said to be convergent if the solution of the scheme tends to the exact solution of the PDE as the grid spacing tends to zero, i.e. $\|E^h\| \to 0 \; as \; h \to 0$

$Consistency + Stability \Rightarrow Convergence$ ([28]

Definition:

The scheme is convergent if as $k \to 0$, $h \to 0$, $\forall \; (t^*, x^*)$, $x_j \to x^*$

$$t_n \to t^*$$

$$\therefore U_j^n \to (t^*, x^*)$$

(consistency implies that the truncation error tends to zero)

So consistency is a necessary but not sufficient condition for convergence

$$U(x_j, t^n) = U_j^n + KT_j^n \; , \quad (26)$$

whereKT_j^n *is the truncation error*

Let $e_j^n = U_j^n - U(t_n, x_j)$ (27)

Where, U_j^n = Numerical solution,

$U(t_n, x_j)$= Exact solution, and

e_j^n = Error

For explicit scheme:

From (1), Taking $\lambda = {}^{k}/_{12h^2}$, we have

$$U_j^{n+1} = U_j^n - a\lambda(-U_{i-2} + 16U_{i-1} - 30U_i + 16U_{i+1} - U_{1+2}) \quad (28)$$

We now have;

$$U(t_{n+1}, x_j) = \text{-U} - U(t_n, x-2h) + 16U(t_n, x-h) - 30U(t_n, x_j) + 16U(t_n, x+h) - U(t_n, x+2h) + KT_j^n \quad (29)$$

Using the differential error, we have;

$$e_j^{n+1} = \quad e_j^n - a\lambda(-e_{j-2}^n + 16e_{j-1}^n - 30e_j^n + 16ej+1n - ej+2n + KTjn \quad (30)$$

$$e_j^{n+1} = \quad e_j^n + a\lambda e_{j-2}^n - 16a\lambda e_{j-1}^n + 30a\lambda e_j^n - 16a\lambda e_{j+1}^n + a\lambda e_{j+2}^n + KT_j^n$$

$$=(1+30a\lambda)e_j^n + a\lambda(e_{j-2}^n + e_{j+2}^n) - 16a\lambda(e_{j-1}^n + ej+1n + KTjn \quad (31)$$

Error analysis with $L_{\infty - Norm}$

We define

$$E^n = max_j\{|e_j^n|\}, \qquad T^n = \max_j\{|e_j^n|\}, \hat{T} = \max_j T^n \quad (32)$$

$$\Rightarrow E^{n+1} = max_j\{|e_j^{n+1}|\}, T^\infty = max_j\{|T_j^n|\}$$

Note: $v_j^0 = u_j^0 \; \forall \; j$, hence,

$$E^0 = 0$$

It then follows from (31)and for some underlying assumptions that;

$$\therefore \max_j|e_j^{n+1}| \leq \max_j|1+30a\lambda|[e_j^n] + a\lambda\max_j[e_{j-2}^n] + a\lambda\max_j[e_{j+2}^n] - 16a\lambda\max_j|e_{j-1}^n| - 16a\lambda\max_j|e_{j+1}^n| + KT_j^n \quad (33)$$

Since the above inequality holds for all j, then;

$$E^{n+1} \leq [1+30a\lambda]E^n + a\lambda E^{n-1} + a\lambda E^{n+1} - 16a\lambda E^{n-1} - 16a\lambda E^{n+1} + KT_j^n \quad (34)$$

Assume $1 + 30a\lambda > 0$, then

$$\Rightarrow E^{n+1} \leq (1+30a\lambda)E^n + a\lambda E^n + a\lambda E^n - 16a\lambda - 16a\lambda E^n + KT^\infty \quad (35)$$

$$E^{n+1} \leq (1+30a\lambda)E^n - 30a\lambda E^n + KT^\propto$$

$$E^{n+1} \leq E^n + 30a\lambda$$

$$E^{n+1} \leq E^n + KT$$

$$E^n \leq E^{n-1} + KT$$

.
.
.

$$E^1 \leq E^0 + nKT$$

since E^0 =0 i.e, no error in the initial data

$\Rightarrow$

$$E^n \leq nKT$$

$\Rightarrow E^n \leq t_f T$

As $h \to 0$, $K \to 0$

$$\Rightarrow E^n \to 0$$

Which proves convergence

IV. CONCLUSION

This study has rigorously examined the consistency, stability, and convergence of the finite difference–based Method of Lines (FD-MoL) using a five-point discretization for solving parabolic partial differential equations, with the heat equation serving as the benchmark model. By discretizing the spatial derivatives through a high-order central difference scheme and applying suitable time integration methods, the resulting semi-discrete formulation was systematically analyzed.

The consistency analysis, grounded in local truncation error evaluation, confirmed that the proposed FD-MoL scheme is first-order accurate in time and second-order accurate in space, ensuring that the discrete equations asymptotically reproduce the continuous PDE as the grid parameters approach zero. Von Neumann stability analysis demonstrated that the implicit formulation of the method satisfies the unconditional stability criterion, with the amplification factor remaining bounded for all admissible wave numbers, independent of the time-step size. Furthermore, convergence was established both theoretically and analytically, showing that the numerical solution approaches the exact solution as the discretization parameters tend to zero, this is in agreement with the Lax equivalence theorem.

The findings underscore the reliability and robustness of the five-point stencil FD-MoL for parabolic PDEs, offering improved spatial resolution while preserving numerical stability across a wide range of time-step selections. This makes the method particularly advantageous for large-scale simulations in Science and Engineering where high spatial accuracy and stability are paramount. Future work may extend the framework to nonlinear PDEs, multi-dimensional systems, and adaptive meshing strategies to further enhance computational efficiency and applicability.

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