International Journal of Advances in Engineering and Management (IJAEM)
Volume 4, Issue 8 Aug. 2022, pp: 973-980 www.ijaem.net ISSN: 2395-5252

Graph Theory as a Mathematical Model: An Introduction with its various applications

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Submitted: 10-08-2022 Revised: 17-08-2022 Accepted: 20-08-2022

ABSTRACT:
Graph Theory has its origin dating back to the 18th century but it has many fascinating modern applications. These applications in turn has offered important stimulus to the development of new areas of applications leading to generalizations of theoretical concepts. Given the wide scope and application of Graph Theory the paper developed by us tries to give a brief overview of the basic concepts and its applications. We used the various terminology of Graph Theory in the paper so that the first learner as well as general enthusiast may get a grasp of the theory.

Keywords: Edges, Vertex, Graph theory, Graph, Network, Application

I. INTRODUCTION:

One essential feature of modern technological age is that it produces vast amounts of information. Facets of our culture -share market, sports' scores, financial transactions, weather statistics, and economic conditions - have features that are countable or measurable in some way thus producing huge amount of information. Graphs have become a means for organizing, displaying, and comparing this information and become useful tools in communication. Graphical presentations are vivid and eye-catching; they focus the reader's attention. Graphic displays have the ability to present data compactly and concisely. They enable the reader to see at a glance fact that might take several paragraphs to describe adequately. As a result Graph theory has evolved as central to all scientific and in case of some non-science based practice or discipline. It is widely used as tools for analyzing and understanding scientific phenomena and presenting non-scientific but empirical deductions. Graph theory started with Euler who was asked to find a nice path across the seven Koningsberg bridges. Scientists, statisticians, engineers, librarians and researchers use graphical representations extensively such that in their absence they fail to accomplish task of explaining the deduced results. But now graph theory is used for finding communities in networks where the objective is to detect the hierarchy of substructures. It is also used for ranking (ordering) of hyperlinks. It is also finding application in GPS to find the shortest path home. The main purpose of graph theory is to facilitate the grasping and understanding of a relational structure represented by a complex organization. Given the wide scope and application of Graph Theory the paper developed by us tries to give a brief overview of the basic concepts and its applications. We used the various terminology of Graph Theory in the paper so that the first learner as well as general enthusiast may get a grasp of the theory. The paper is developed with an Introduction, then Section 1 giving the details of Graph Theory, section 2 giving the History, section 3 depicts the details of applications of Graph Theory and finally the paper is summed up with the Conclusion.

What is Graph Theory?

In Computer Science and mathematics, graph theory is the comprehensive study of graph which is the made up of nodes or vertices (also called points) and those vertices are connected by the edges (also called link or lines). A graph contains a particular shape which is distinguished by their placement.

A graph G = (V,E) is a pair of vertices (or nodes) V and a set of edges E, assumed finite i.e. |V| = n and |E| = m, here V(G) = {v1,v2,...,v5}and E(G) = {e1,e2,...,e6}. An edge ek = (vi,vj) is incident with the vertices vi and vj.

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Fundamentals of Graph Theory:
Before going into our main topic, we will briefly elaborate some basic jargons of the graph theory. Those are as follows:

Point:
A point is a particular position regarding one-dimensional, two-dimensional, or three-dimensional space. It can be described with dot 'a' and is usually denoted by an alphabet.
Example

Line:
A Line is a relationship between two points and usually it can be denoted by a solid line.
Example

Vertex or node:
A vertex or node is a point where different lines meet. Just like a point, it is also denoted by an alphabet i.e. 'V'.
According to the degree of vertex there are special two types of vertex. Those are:

Pendent Vertex:
A vertex with degree one is called a pendent vertex i.e. deg(a) = 1, as there is 1 edge formed at vertex 'a'. So 'a' is a pendent vertex.
Example

Isolated Vertex:
A vertex with degree zero is called an isolated vertex i.e. deg(a) = 0, as there are 0 edges formed at vertex 'a'. So 'a' is an isolated vertex.
Example

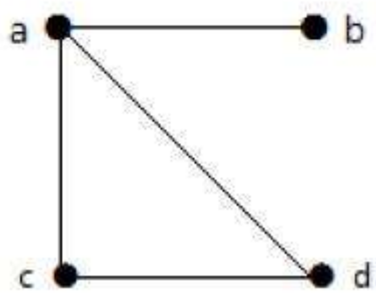
Edge:
An edge is the mathematical terminology of a line that joins two vertices or node. An edge cannot be made without a vertex. There must be two vertex to form a edge i.e. a starting vertex and an ending vertex. It is also denoted by an alphabet i.e. E.
Example:
Here, 'a' and 'b' are the two vertices and the link between them is called an edge. Here a is the starting vertex and b is an ending vertex.

Graph:
A graph represents a network which consists of set of some vertices or nodes. These nodes are interconnected with each other based upon some relation.
Mathematically we can represent a graph with an alphabet 'G'

$$G = (V, E)$$

Where, V = set of all vertices or nodes
E = set of all edges.
Example 1

Here, ab, ac, cd, and bd are the edges of the graph. Similarly, a, b, c, and d are the vertices or nodes of the graph.
Example 2



In this example 2, there are total 4 vertices a, b, c, and d, and 4 edges named, ab, ac, ad, and cd.

Loop:
If an edge is drawn from vertex to itself, then it can be termed as a loop.
Example
In the following graph, V is a vertex for which it has an edge (V, V) forming a loop.

Undirected Graph:
When the edges do not have any particular direction from one node or vertex to another one then it is called as undirected graph. Here the edge normally denoted by a straight line.

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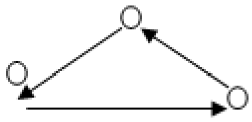
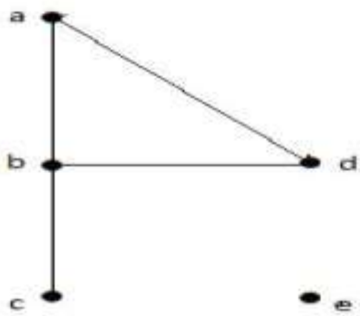
Directed Graph:
When there is a particular direction of edges from one node or vertex to another then it is termed as directed graph. Here the edge normally denoted by an arrow.



Degree of Vertex:
It is a total number of vertices or nodes adjacent to a vertex V. It is denoted with deg(V).

Degree of Vertex in an Undirected Graph:
There is no directed edge in an undirected graph. The following example shows the degree of Vertex in an Undirected Graph.
In the following Undirected Graph,



- deg(a) = 2, as there are 2 edges meeting at vertex ‘a’.
- deg(b) = 3, as there are 3 edges meeting at vertex ‘b’.
- deg(c) = 1, as there is 1 edge formed at vertex ‘c’. So ‘c’ is a pendent vertex.
- deg(d) = 2, as there are 2 edges meeting at vertex ‘d’.
- deg(e) = 0, as there are 0 edges formed at vertex ‘e’. So ‘e’ is an isolated vertex.

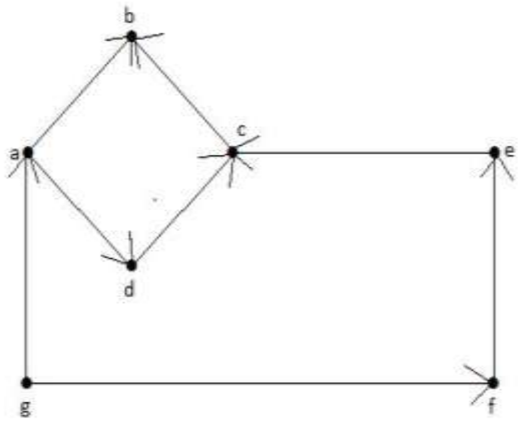
Degree of Vertex in a Directed Graph:
In directed graph each and every vertex has an outdegree and an indegree.

Indegree of a Graph:
The number of edges which are coming into the vertex V that is called as indegree of vertex V. Indegree of vertex V is denoted by deg−(V).

Outdegree of a Graph:
The number of edges which are going out from the vertex V that is called as outdegree of vertex V. Outdegree of vertex V is denoted by deg+(V).

Example

Take a look at the following directed graph. Vertex ‘a’ has two edges, ‘ad’ and ‘ab’, which are going outwards. Hence its outdegree is 2. Similarly, there is an edge ‘ga’, coming towards vertex ‘a’. Hence the indegree of ‘a’ is 1.



The following table shows the indegree and outdegree of other vertices –

Vertex	Outdegree	Indegree
a	2	1
b	0	2
c	1	2
d	1	1
e	1	1
f	1	1
g	2	0

Adjacency:

- ✓ If two vertices or nodes are consider to be adjacent, if there is an edge between the two nodes or vertices. The adjacency of nodes or vertices is supported by the single edge that is connecting those two nodes or vertices.
- ✓ If two edges are consider to be adjacent, if there is a common node or vertex between the two edges. The adjacency of edges is supported by the single nodes or vertex that is connecting two edges.

Example

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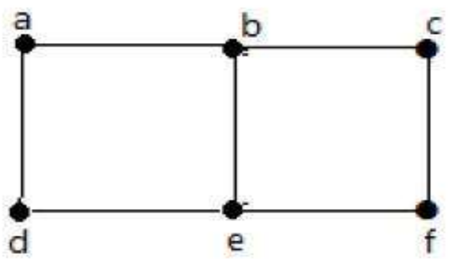


In the above graph −

✓ ‘b’ and ‘c’ are the adjacent vertices, as there is a common edge ‘bc’ between them.

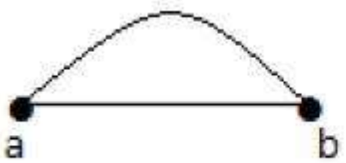
✓ ‘b’ and ‘e’ are the adjacent vertices, as there is a common edge ‘be’ between them.

✓ ‘bc’ and ‘cf’ are the adjacent edges, as there is a common vertex ‘c’ between them.

✓ ‘cf’ and ‘fe’ are the adjacent edges, as there is a common vertex ‘f’ between them.

Parallel Edges:

If two nodes or vertices are connected by more than one edges in a graph, then those edges are termed as parallel edges.



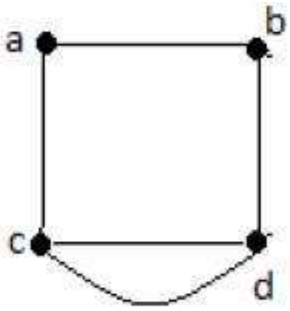
Here, ‘a’ and ‘b’ are the pair of nodes which are connected by two edges i.e. ‘ab’ and ‘ab’. That’s why it is termed as Parallel edges.

Multi Graph:

A graph consists of parallel edges can be known as a Multigraph.

Example

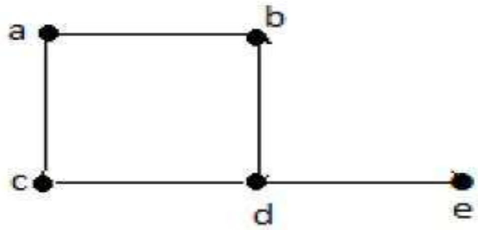
In the following graph, there are five edges ‘ab’, ‘ac’, ‘cd’, ‘cd’, and ‘bd’. Since ‘c’ and ‘d’ have two parallel edges between them, so it a Multigraph.



Degree Sequence of a Graph:

If we arrange the degrees of all nodes or vertices in a graph in descending or ascending order, then the sequence obtained is known as the degree sequence of the graph.

Example



In the following table, for the vertices {d, a, b, c, e}, the degree sequence is {3, 2, 2, 2, 1}.

Vertex or node	A	b	c	d	e
Connecting to	b,c	a,d	a,d	c,b,e	d
Degree	2	2	2	3	1

Types of Graph:

There are various types of graphs according to the number of edges, interconnectivity, number of vertices, and their overall structure.These are as follows:

Null Graph:

A null graph is that kind of graph which doesn’t have any edges.

Example

In the above graph, there are two vertices or nodes named ‘a’ and ‘b’, but there are no edges among them. So it can be called as a Null Graph.

Trivial Graph:

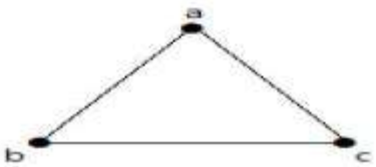
A graph having only one vertex or node is called a Trivial Graph.

Example

In the above shown graph, there is only one node ‘a’ with no other edges. That’s why it is called astrivial graph.

Undirected Graph:

When the edges do not have any particular direction from one node or vertex to another one then it is called as undirected graph. Here the edge normally denoted by **a straight line**.



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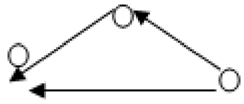
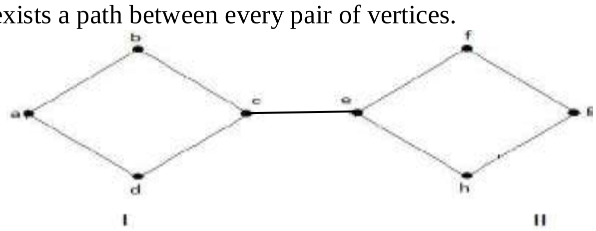
Directed Graph:
When there is a particular direction of edges from one node or vertex to another then it is termed as directed graph. Here the edge normally denoted by **an arrow**.



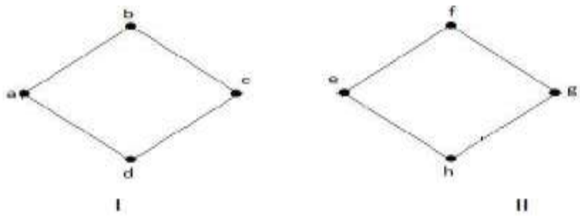
Simple Graph:
When there is a absence of loops and parallel edges in any particular graph then it is called as Simple graph.

In the above graphs there are no loops and parallel edges so it can be considered as a simple graph.

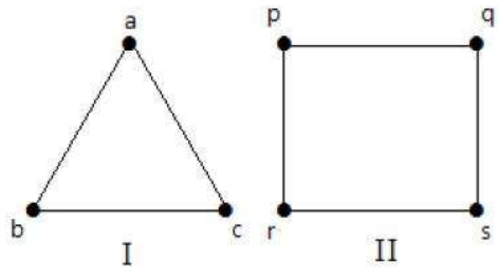
Connected Graph:
A graph is considered to be as connected if there exists a path between every pair of vertices.



Disconnected Graph:
A graph is considered as disconnected, if there does not contain at least two connected vertices.

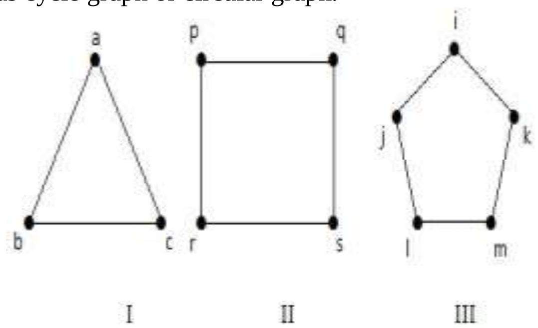


Regular Graph:
If all the vertices or nodes in a graph have the same degree then that kind of graph is termed as Regular graph.
Example
In the graphs shows in the next page, all the vertices have the same degree. So these graphs are called regular graphs.



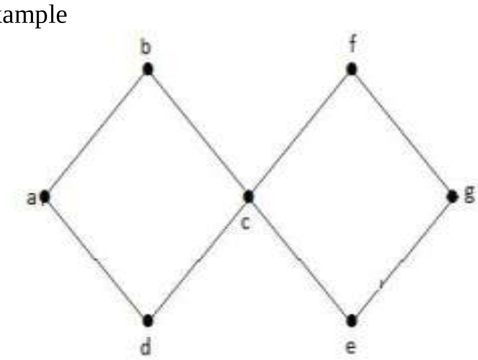
In both the graphs, all the vertices have degree 2. They are called 2-Regular Graphs.

Cycle graph or circular graph:
When a graph that consists of a single cycle, or in other words, some number of vertices or nodes (at least 3) connected in a closed chain then it is called as cycle graph or circular graph.



Wheel Graph:
A wheel graph is obtained from a cycle graph by adding a new vertex. That new vertex is called a Hub which is connected to all the vertices.

Cyclic Graph:
When there is at least one cycle in a particular graph then it is called as cyclic graph.
Example



In the above graph, we have two cycles a-b-c-d-a and c-f-g-e-c. So it is called a cyclic graph.

Acyclic Graph:
A graph having no cycles is called an acyclic graph.
Example

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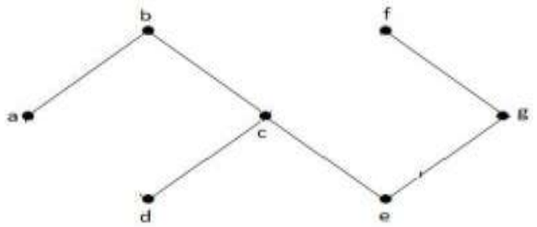

In the above example graph, we do not have any cycles. Hence it is a non-cyclic graph.
Apart from the above different types of graphs there are also many other types of graphs. These are as follows:

- Bipartite Graph
- Complete Bipartite Graph
- Star Graph
- Complement of a Graph

History of graph theory:

The primary concept of graph theory was first introduced by the Swiss mathematician Leonhard Euler in the 18th century. The Konigsberg Bridge Problem is perhaps the best known example in graph theory. It was a long-standing problem until solved by Euler in 1736 by means of a graph. Euler wrote the first research paper in graph theory and then became the originator of the theory of graphs. The German city of Konigsberg, presently known as Kaliningrad, Russia is situated on Pregolya river. The geographical layout of this city is composed of four main lands which are connected by total seven bridges. The main problem is that was it possible to visit in each and every city in such a way as to cross every bridge only for once. So to solve this problem Euler drew a graph to overcome the relevant constraints and this graph is the first visual representation of the modern graph. The pictorial representation of the problem (Fig.1) and the graph (Fig.2) are as follows:

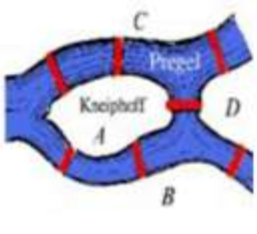

Fig.1

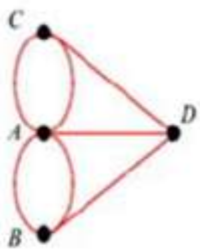

Fig.2

Application of the Graph Theory:

The term 'Application' means the implementation of any theory, model, design, policy or execution of a plan. As our topic of discussion is the graph theory, so it has many fascinating modern applications in various body of knowledge. Graph Theory is the study of relationships of vertex and edges. It is also known as network science. It has evolved tremendously in the last decade. So some popular applications of the graph theory are as follows:

1. Social Science:

In Social Science graph theory is widely applied to represent quantitative data in easily comprehensible format. Bar Graphs and Line graphs represent quantities in terms of spatial extent, whereas pie charts represent quantities in terms of area or angle size. In principle, it is possible to represent quantitative information with a number of other possible visual features, such as color saturation. In Economics graph theory finds application in the depiction of sequential games where a connected and acyclical graph is used in which each vertex represents a decision point and each node represents an action of the player to which the above vertex was assigned to.

2. Linguistics:

Linguistics is the scientific study of language. There are many uses of graph theory which can be applied in the field of linguistics like syntax and compositional semantics, lexical semantics, phonology and morphology. In terminological tree of a language and in grammar of a language uses graphs.

3. Physics and Chemistry:

Graph theory is also used to study molecules in physics and chemistry. In condensed matter physics, the three-dimensional structure of complicated simulated atomic structures can be studied by graph theory. In statistical physics graph theory can also be applied.Chemical graph theory uses the molecular graph as a means to model molecules.

4. Biology:

Graph theory is also useful in Biology where a vertex can represent regions where certain species exist (or inhabit) and the edges represent migration paths or movement between the regions. This information is important when looking at breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species.

5. Genetics:

Graph theory is also used in Genetics where the nervous systems can be seen as a graph, where the nodes are neurons and the edges are the connections between them.

6. Mathematics:

In mathematics, graphs are useful in geometry, knot theory, Algebraic graph theory andmany more.

7. Engineering:

The application of graph theory also used in the different fields of engineering like electrical engineering, civil engineering, computer science, computer network etc.

8. Library and Information Science:

Now a day's graphs theory are becoming the new data models for the librarians. There are many practical applications of graph theory in the field of digital libraries.

9. Social network:

The concept of graph theory also widely used in the field of social networking sites like facebook, twitter etc.

II. CONCLUSION:

Graph Theory has since its development in the 18th century has been widely applied and used in a variety of fields. The power of graphs comes from the topological character of the lines that articulate relations between the variables indicated on the axes. The lines encode continuous change and are therefore suited to represent the dynamic nature of physical phenomena. Linguistic representations, on the other hand, divide the world into objects and classes of objects: Verbal representation is typological in character. Graphs, consisting of combinations of topological and typological features, have specificity (what they cannot leave un said about the observed situation) that aids in their use for constructing logical arguments. Hence Graph Theory with its multi-faceted representation and characterization has been widely used and applied. Thus, it goes beyond any saying that Graph theory since its development has been very useful in representing complex problems.

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