International Journal of Advances in Engineering and Management (IJAEM)
Volume 3, Issue 7 July 2021, pp: 1902-1909 www.ijaem.net ISSN: 2395-5252

On the Jacobs thaland Jacobsthal Lucas Sequences at Negative Indices

S. Uygun
Department of Mathematics, Scienceand Art Faculty, Gaziantep University,Campus, Gaziantep, Turkey

Submitted: 01-07-2021 Revised: 13-07-2021 Accepted: 16-07-2021

TheJacobsthalnumbers j_n areterms of thesequence $\{0,1,1,3,5,11,...\}$,definedbytherecurrencerelation $j_n = j_{n-1} + 2j_{n-2}$ beginningwiththevalues $j_0 = 0, \ j_1 = 1$. Similarly, theJacobsthalLucasnumbers c_n areterms of thesequence $\{2,1,5,7,17,...\}$ definedbytherecurrencerelation $c_n = c_{n-1} + 2c_{n-2}$ beginningwiththevalues $(c_0 = 2, \ c_1 = 1)$ in [1]. Andtherelationbetweenthesesequencesaregiven as JacobsthalLucassequence as

$$c_n = 2j_{n-1} + j_{n+1} \quad (1)$$

$$9j_n = 2c_{n-1} + c_{n+1} \quad (2)$$

TherearemanypapersaboutJacobsthalandJacobsthal Lucasnumbers in thelastdecadeyears. Forexampleyou can canfind in thereferences [1-6]. TheauthorsgaveformulasforFibonacciandLucassequences at negativeindices in [13]. SimilarlyDaşdemirextendedMersenne, JacobsthalandJacobsthalLucastotheirtermswithnegativesubscriptsandfoundsomeimportantrelationships in [14]. JacobsthalandJacobsthalLucasnumbers at negativeindicesareobtainedbyusingthefollowingequalities:

$$c_{-n} = \frac{(-1)^n}{2^n} c_n \quad (3)$$

$$j_{-n} = \frac{(-1)^{n+1}}{2^n} j_n \quad (4)$$

First Jacobsthalnumbers at negativeindicesare $j_{-1}=1/2$, $j_{-2}=(-1)/4$, $j_{-3}=3/8$, $j_{-4}=-5/16$, $j_{-5}=11/32$, $j_{-6}=-21/64$. First JacobsthalLucasnumbers at negativeindicesare $c_{-1}=-1/2$, $c_{-2}=5/4$, $c_{-3}=-7/8$, $c_{-4}=17/16$, $c_{-5}=-31/32$, $j_{-6}=65/64$.

Therearemanygeneralizations on thesesequences. Forexample, (s,t)-Jacobsthaland (s,t)-JacobsthalLucassequencesaredefinedbyusingthefollowingrecurrencerelations

$$j_n(s,t) = j_{n-1}(s,t) + 2j_{n-2}(s,t), \ j_0(s,t) = 0, \ j_1(s,t) = 1$$

$$c_n(s,t) = c_{n-1}(s,t) + 2c_{n-2}(s,t), \ c_0(s,t) = 2, \ c_1(s,t) = s$$

where $s>0$, $t\neq 0$ and $s^2+8t>0$ respectively in [2]. (s,t)-Jacobsthaland (s,t)-JacobsthalLucassequences at negativeindices can be given as

$$j_{-n}(s,t) = \frac{sj_{-n+1}(s,t) - j_{-n+2}(s,t)}{2t}$$

$$c_{-n}(s,t) = \frac{sc_{-n+1}(s,t) - c_{-n+2}(s,t)}{2t}$$

Matrixalgebraplays an important role in thetheory of specialintegersequences. So, in [7], Williams studiedthenthpower of a 2x2 matrix. BergumandHoggattinvestigatedthesumsandproductsforrecurringsequences in [8]. Laughlin, studiedcombinatorialidentitiesderivingfromthenthpower of somematrices in [9,10]. ThenBelbachirfoundlinearrecurrentsequencesandpowers of a squarematrix in [11]. Andtheauthorsderivedcombinatorialidentitiesbyusingthetrace, the determinant andthenthpower of a specialmatrixwhoseentriesaregeneralizedFibonacci andLucasnumbers in [12].
Binetformulaenables us tostateJacobsthalandJacobsthalLucasnumberseasily. It can be clearlyobtainedfromtherootsr$_1$ =2andr$_2$ =-1 of

International Journal of Advances in Engineering and Management (IJAEM)
Volume 3, Issue 7 July 2021, pp: 1902-1909 www.ijaem.net ISSN: 2395-5252

characteristicequationof therecurrencerelation as the form $x^2=x+2$. TheBinetformulasforJacobsthalandJacobsthalLucas numbersnumbersaregivenby

$$j_n = \frac{r_1^n - r_2^n}{r_1 - r_2}$$

$$c_n = r_1^n + r_2^n$$

In [7], Williams, gave a well-knownformulathatif $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,then

$$A^n = \begin{cases} \dfrac{x_1^n - x_2^n}{x_1 - x_2} A - \dfrac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2} I_2, & x_1 \neq x_2 \\ n x^{n-1} A - (n-1) \det(A) x^{n-2}, & x_1 = x_2 \end{cases}$$

x_1 ,x_2 beingtheroots of theassociatedcharacteristicequation of thematrixA

$$r^2 - (a+d)r + \det(A) = 0.$$

Laughlin, in [9,10] gaveif A is a 2x2 matrix as $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ thenthenthpower of A is givenby

$$A^n = \begin{bmatrix} x_n - d x_{n-1} & b x_{n-1} \\ c x_{n-1} & x_n - a x_{n-1} \end{bmatrix}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (trA)^{n-2i} (-\det A)^i$.

Proposition 1: In [14], JacobsthalandJacobsthalLucasnumbers at negativeindicesaresatisfiedthefollowingidentities:

$$j_{-(n+1)} j_{-(n-1)} - j^2_{-n} = (-1)^{n-1} 2^{n-1} \qquad (6)$$

$$c_{-(n+1)} c_{-(n-1)} - c^2_{-n} = (-1)^{n-1} 2^{n-1} \qquad (7)$$

$$2 j_{-(n+1)} + j_{-(n-1)} = c_{-n} \qquad (8)$$

$$2 c_{-(n+1)} + c_{-(n-1)} = 9 j_{-n} \qquad (9)$$

Proposition 2: In [14], D'OcagnepropertiesforJacobsthalandJacobsthalLucasnumbers at negativeindicesaredemonstratedby

$$j_{-(m+1)} j_{-n} - j_{-m} j_{-(n+1)} = (-1)^m 2^{-m-1} j_{-(n-m)} \qquad (10)$$

$$c_{-(m+1)} c_{-n} - c_{-m} c_{-(n+1)} = 9(-1)^n 2^{-n-1} j_{-(m-n)} \qquad (11)$$

$$c_{-r} j_{-(n-1)} - c_{-(r-1)} j_{-n} = (-1)^n 2^{-n} c_{n-r} \qquad (12)$$

$$j_{-r} c_{-(n-1)} - j_{-(r-1)} c_{-n} = (-1)^n 2^{-n} c_{n-r} \qquad (13)$$

Theorem 3: Let us consider a specialmatrix as follows

$$J = \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix} \qquad (14)$$

Thenthpower of J is calculatedbyusingJacobsthalsequencesandJacobsthalLucassequences at negativeindices as

$$J^{n}=6^{n}\begin{bmatrix}2j_{-(n+1)} & j_{-n}\\ 2j_{-n} & j_{-(n-1)}\end{bmatrix}, \qquad n \ is \ even \qquad (15)$$

$$J^{n}=(6^{n}/3)\begin{bmatrix}2c_{-(n+1)} & c_{-n}\\ 2c_{-n} & c_{-(n-1)}\end{bmatrix}, \qquad n \ is \ odd \qquad (16)$$

Proof: Forn=0, thestatement is true. Let us it is trueforallk≤n, and n is even. Thenforn=k+1, weinvestigatethetruthness of theclaim.

$$J^{n+1}=6^{n}\begin{bmatrix}2j_{-(n+1)} & j_{-n}\\ 2j_{-n} & j_{-(n-1)}\end{bmatrix}\begin{bmatrix}5 & -1\\ -2 & 4\end{bmatrix}$$

Forthe (1,1) element of J^{n+1}

$$J^{n}=6^{n}\left[10\frac{(-1)^{n+2}}{2^{n+1}}j_{(n+1)}-2\frac{(-1)^{n+1}}{2^{n+1}}j_{n}\right]=2.6^{n}\left[\frac{(-1)^{n+2}}{2^{n+1}}(5j_{(n+1)}+2j_{n})\right]$$
$$=2.6^{n}\left[\frac{(-1)^{n+2}}{2^{n+1}}c_{n+2}\right]=2.6^{n}(2c_{-(n+2)})$$

Theotherelements is obtainedbyusing a similarway. Nowassumen is odd

$$J^{n+1}=(6^{n}/3)\begin{bmatrix}2c_{-(n+1)} & c_{-n}\\ 2c_{-n} & c_{-(n-1)}\end{bmatrix}\begin{bmatrix}5 & -1\\ -2 & 4\end{bmatrix}$$

Forthe (1,1) element of J^{n+1}

$$(6^{n}/3)\left[10\frac{(-1)^{n+1}}{2^{n+1}}c_{n+1}-2\frac{(-1)^{n}}{2^{n}}c_{n}\right]=2^{n+1}.3^{n-1}\left[\frac{(-1)^{n+1}}{2^{n+1}}(5c_{n+1}+2c_{n})\right]$$
$$=2^{n+1}.3^{n-1}\left[\frac{(-1)^{n+1}}{2^{n+1}}9j_{n+2}\right]=2.6^{n+1}j_{-(n+2)}.$$

Theotherelements is obtainedbyusing a similarway.

Theorem 4: Forpositiveintegersn, explicitclosed form expressionsforJacobsthalsequencesandJacobsthalLucassequences at negativeindicesaregiven as

International Journal of Advances in Engineering and Management (IJAEM)
Volume 3, Issue 7 July 2021, pp: 1902-1909 www.ijaem.net ISSN: 2395-5252

$$j_{-n}=\sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n-1-k}{k}3^{n-2-2k}2^{k-n}(-1)^{k+1}, \quad if\ \ n\ \ is\ even$$

$$c_{-n}=\sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n-1-k}{k}9^{n-1-2k}2^{k-n}(-1)^{k+1}, \quad if\ \ n\ \ is\ odd$$

Proof: Byusing (5-9), thenthpower of J is

$$J^n=6^n\begin{bmatrix} x_n-4x_{n-1} & -x_{n-1} \\ -2x_{n-1} & x_n-5x_{n-1}\end{bmatrix}$$

where $x_n=\sum_{i=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n-i}{i}9^{n-2i}(-18)^i=\sum_{i=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n-i}{i}9^{n-i}(-2)^i$. Bytheequality of correspondingentries of (1,2), ifn is an evennumber,and $j_{-n}=-\sum_{i=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n-1-i}{i}9^{\frac{n}{2}-1-i}2^{i-n}(-1)^{i+1}$ ifn is oddnumber $c_{-n}=\sum_{i=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n-1-i}{i}9^{\frac{n-1}{2}-i}2^{i-n}(-1)^{i+1}$.

Theorem 5: Forn,kpositivenumbers

$$j_{-nk}=j_{-n}c_{-n}^{k-1}\sum_{i=0}^{\left\lfloor \frac{k-1}{2}\right\rfloor}\binom{k-1-i}{i}c_n^{-2i}2^{ni}(-1)^i, \quad if\ \ n,k\ \ are\ even$$

$$c_{-nk}=j_{-n}^{k-1}c_{-n}3^{2k-2}\sum_{i=0}^{\left\lfloor \frac{k-1}{2}\right\rfloor}\binom{k-1-i}{i}9^{-i}2^{-ni}(-1)^{i+1}j_{-n}^{-2i}, \quad if\ \ n,k\ \ are\ odd$$

$$j_{-nk}=3^{k+1}j_{-n}^{k-1}c_{-n}\sum_{i=0}^{\left\lfloor \frac{k-1}{2}\right\rfloor}\binom{k-1-i}{i}j_{-n}^{-2i}2^{-ni}9^{-i}(-1)^i, \quad if\ \ n\,is\,odd\,,k\ \ is\,even$$

$$j_{-nk}=j_{-n}\sum_{i=0}^{\left\lfloor \frac{k-1}{2}\right\rfloor}\binom{k-1-i}{i}c_{-n}^{k-1-2i}2^{-ni}9^{-i}(-1)^i, \quad if\ \ k\,is\,odd\,,n\ \ is\,even$$

Proof: Ifn is an evennumber. thekthpower of thematrixJ^n is demonstratedby

$$J^{nk}=\begin{bmatrix} x_k-6^n j_{-(n-1)}x_{k-1} & 6^n j_{-n}x_{k-1} \\ 2.6^n j_{-n}x_{k-1} & x_k-2^{n+1}9^{\frac{n-1}{2}} j_{-(n+1)}x_{k-1} \end{bmatrix}$$

where $x_k=\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}\binom{k-i}{i}((-18)^n)^i c_{-n}^{k-2i}$ since by (6), (8) as $det(J^n)=2^{2n+1}9^n\left[j_{-(n+1)}j_{-(n-1)}-j_{-n}^2\right]=(-18)^n$ and $tr(J^n)=2j_{-(n+1)}+j_{-(n-1)}=c_{-n}$. In (15) thesubstitutionfor-n →-nk,,givesthenk.th power of J. Bytheequality of correspondingentries of thematrices, thedesiredresult is obtained.
Ifn is an oddnumber

$$J^{nk}=\begin{bmatrix} x_k-2^n 3^{n-1}c_{-(n-1)}x_{k-1} & 2^{nk}3^{n-1}c_{-n}x_{k-1} \\ 2^{n+1}3^{n-1}c_{-n}x_{k-1} & x_k-2^{n+1}3^{n-1}c_{-(n+1)}x_{k-1} \end{bmatrix}$$

where $x_k=\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}\binom{k-i}{i}(-18)^{ni}9^{k-2i}c_{-n}^{k-2i}$ because of (7), (9), $det(J^n)=2^{2n+1}9^{n-1}\left[c_{-(n+1)}c_{-(n-1)}-c_{-n}^2\right]=-(-18)^n$ and $tr(J^n)=2c_{-(n+1)}+c_{-(n-1)}=9j_{-n}$.
Theresult is obtainedbytheequality of thematrices.

Corollary 6: Byusingmatrixproduct, thefollowingidentitiesaredenoted

$$2j_{-(n+1)}j_{-m}+j_{-n}j_{-(m-1)}=j_{-(n+m)},\quad m,n\ are\ even$$

$$2c_{-(n+1)}c_{-m}+c_{-n}c_{-(m-1)}=9j_{-(n+m)},\quad m,n\ are\ odd$$

$$2j_{-(n+1)}c_{-m}+j_{-n}c_{-(m-1)}=c_{-(n+m)},\quad m\ is\ odd, n\ is\ even$$

$$2j_{-(m+1)}c_{-n}+j_{-m}c_{-(n-1)}=c_{-(n+m)},\quad n\ is\ odd, m\ is\ even$$

Proof: ,Ifm, nareevennumbers, thenm+nis alsoevennumber. ByTheorem 3, it is satisfied:

$$J^m J^n=6^{m+n}\begin{bmatrix} 2j_{-(m+1)} & j_{-m} \\ 2j_{-m} & j_{-(m-1)} \end{bmatrix}\begin{bmatrix} 2j_{-(n+1)} & j_{-n} \\ 2j_{-n} & j_{-(n-1)} \end{bmatrix}$$

$$=J^{m+n}=6^{m+n}\begin{bmatrix} 2j_{-(m+n+1)} & j_{-(m+n)} \\ 2j_{-(m+n)} & j_{-(m+n-1)} \end{bmatrix}$$

Bytheequality of (2,1) elements of matrices, theresult is obtained. Theotherresultsarealsofoundbysimilarway.

Corollary 7: Byusingmatrixproduct

International Journal of Advances in Engineering and Management (IJAEM)
Volume 3, Issue 7 July 2021, pp: 1902-1909 www.ijaem.net ISSN: 2395-5252

$$-2j_{-(m+1)}j_{-n}+j_{-m}j_{-(n+1)}=2^{-n}j_{-(m-n)},\quad m,n \text{ are even}$$
$$-2c_{-(m+1)}c_{-n}+c_{-m}c_{-(n+1)}=9.2^{-n}j_{-(m-n)},\quad m,n \text{ are odd}$$
$$-2j_{-(m+1)}c_{-n}+j_{-m}c_{-(n+1)}=\frac{2^{-n}}{9}c_{-(m-n)},\quad n \text{ is odd}, m \text{ is even}$$
$$-2c_{-(m+1)}j_{-n}+c_{-m}j_{-(n+1)}=2^{-n}c_{-(m-n)},\quad m \text{ is odd}, n \text{ is even}$$

Proof: Ifm, nareevennumbers, thenm+n is alsoevennumber. ByTheorem 3, it is satisfied:

$$J^mJ^{-n}=6^m\begin{bmatrix}2j_{-(m+1)} & j_{-m}\\ 2j_{-m} & j_{-(m-1)}\end{bmatrix}\frac{6^n}{(-18)^n}\begin{bmatrix}2j_{-(n-1)} & -j_{-n}\\ -2j_{-n} & j_{-(n+1)}\end{bmatrix}$$
$$=J^{m-n}=6^{m-n}\begin{bmatrix}2j_{-(m-n+1)} & j_{-(m-n)}\\ 2j_{-(m-n)} & j_{-(m-n-1)}\end{bmatrix}$$

Bytheequality of (2,1) elements of matrices, theresult is obtained. Theotherresultsarealsofoundsimilarly.

Theorem 8: Let us assumethatn,r,karepositiveintegers, thefollowingidentitiesaresatisfied:

Fork, n,reven

$$j_{-(nk+r)}=\sum_{i=0}^{\lfloor k/2\rfloor}\binom{k-i}{i}(-1)^i(2)^{-ni}(c_{-n})^{k-2i}\left(j_{-r}-\frac{k-2i}{k-i}\frac{j_{-(n-r)}}{2^rc_{-n}}\right),$$

For k, n, r odd

$$j_{-(nk+r)}=\sum_{i=0}^{\lfloor k/2\rfloor}\binom{k-i}{i}(-1)^i(2)^{-ni}(j_{-n})^{k-2i}\left(c_{-r}-\frac{k-2i}{k-i}\frac{j_{-(n-r)}}{2^rj_{-n}}\right),$$

For n odd, k, r even

$$j_{-(nk+r)}=\sum_{i=0}^{\lfloor k/2\rfloor}\binom{k-i}{i}(-1)^i(2)^{-ni}(3j_{-n})^{k-2i}\left(j_{-r}-\frac{k-2i}{k-i}\frac{c_{-(n-r)}}{9j_{-n}}\right),$$

For n, r even, k odd

$$j_{-(nk+r)}=\sum_{i=0}^{\lfloor k/2\rfloor}\binom{k-i}{i}(-1)^i(2)^{-ni}(c_{-n})^{k-2i}\left(j_{-r}-\frac{k-2i}{k-i}\frac{j_{-(n-r)}}{2^rc_{-n}}\right),$$

For n, r odd, k even

$$c_{-(nk+r)}=\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}\binom{k-i}{i}(-1)^i(2)^{-ni}(3c_{-n})^{k-2i}\left(c_{-r}-\frac{k-2i}{k-i}\frac{j_{-(n-r)}}{2^r j_{-n}}\right),$$

For k, r odd, n even

$$c_{-(nk+r)}=\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}\binom{k-i}{i}(-1)^i(2)^{-ni}(c_{-n})^{k-2i}\left(c_{-r}-\frac{k-2i}{k-i}\frac{c_{-(r-n)}}{c_n}\right),$$

For k, n odd, r even

$$c_{-(nk+r)}=\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}\binom{k-i}{i}(-1)^i(2)^{-ni}9^{k-i}(j_{-n})^{k-2i}\left(j_{-r}-\frac{k-2i}{k-i}\frac{c_{-(r-n)}}{9j_n}\right),$$

For k, n even, r odd

$$c_{-(nk+r)}=\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}\binom{k-i}{i}(-1)^i(2)^{-ni}(c_{-n})^{k-2i}\left(c_{-r}-\frac{k-2i}{k-i}\frac{c_{-(r-n)}}{c_n}\right)$$

Proof: Assumethatn,k,revenintegers, thennk+ris even. ByusingTheorem 3, it is obtainedthat

$$J^{nk+r}=6^{nk+r}\begin{pmatrix}2j_{-(nk+r+1)} & j_{-(nk+r)}\\ 2j_{-(nk+r)} & 2j_{-(nk+r-1)}\end{pmatrix}$$

Thenby(5), it is writtenthat

$$J^{nk+r}=(J^n)^k J^r=6^r\begin{pmatrix}y_k-6^n j_{-(n-1)}y_{k-1} & 6^n j_{-n}y_{k-1}\\ 2.6^n j_{-n}y_{k-1} & y_k-2.6^n j_{-(n+1)}y_{k-1}\end{pmatrix}\begin{pmatrix}2j_{-(r+1)} & j-r\\ 2j-r & j_{-(r-1)}\end{pmatrix}$$

where $y_k=\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}\binom{k-i}{i}(6^n c_{-n})^{k-2i}\left(-(-18)^n\right)^i$. Bytheequality of matrices

$$6^r\left[\left(y_k-6^n j_{-(n-1)}y_{k-1}\right)j_{-r}+6^n j_{-n}y_{k-1}j_{-(r-1)}\right]=6^{nk+r}j_{-(nk+r)}$$

Aftersomealgebraicoperation

$$6^{nk} j_{-(nk+r)} = j-r\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}\binom{k-i}{i}\left(6^n c_{-n}\right)^{k-2i}\left(-(-18)^n\right)^i$$
$$-6^n j_{-r} j_{-(n-1)}\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor}\binom{k-1-i}{i}\left(6^n c_{-n}\right)^{k-1-2i}\left(-(-18)^n\right)^i$$
$$+6^n j_{-n} j_{-(r-1)}\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor}\binom{k-1-i}{i}\left(6^n c_{-n}\right)^{k-1-2i}\left(-(-18)^n\right)^i$$
$$=\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}\binom{k-i}{i}\left(6^n c_{-n}\right)^{k-2i}\left(-(-18)^n\right)^i$$
$$\cdot\left[j_{-r}-\frac{k-2i}{k-i}\frac{1}{c_{-n}}\left(j_{-n} j_{-(r-1)}-j_{-(n-1)} j_{-r}\right)\right]$$
$$=\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}\binom{k-i}{i}(-1)^i 2^{-ni}\left(c_{-n}\right)^{k-2i}\left[j_{-r}-\frac{k-2i}{k-i}\frac{1}{c_{-n}}\frac{j_{-(n-r)}}{2^r}\right]$$

Fortheotherproofs a similarway is used.

REFERENCES

[1]. A. F. Horadam, (1996), Jacobsthalrepresentationnumbers, TheFibonacciQuarterly., 34(1), 40-54.

[2]. S. Uygun, (2015), The (s,t)-Jacobsthaland (s,t)-JacobsthalLucassequences, Applied Mathematical Sciences, 70(9), 3467-3476.

[3]. F. Köken, D. Bozkurt, On theJacobsthalNumbersbyMatrixMethods, Int. J. Contemp. Math Sciences, 3, (13), (2008), 605-614.

[4]. S. Uygun, A. Erdoğdu, Binominaltransforms of k-Jacobsthalsequences, Journal of Mathematical andComputationalScience, 7(6), (2017), 1100-1114.

[5]. Yalçın F., Tasçı D., Duman E., Generalization of Vieta-JacobsthalandVieta-Jacobsthal-Lucaspolynomials, Mathematical Communications, 20, 2015, 241--251.

[6]. S. Uygun, E. Owusu, A New Generalization of JacobsthalNumbers (Bi-PeriodicJacobsthalSequences), Journal of Mathematical Analysis 7(5), 2016, 28-39.

[7]. K. S. Williams, Thenthpower of a 2x2 matrix (in notes), Math. Magazine, 65(5), 336.

[8]. G. E. Bergum, V. E. HoggattJr.,Sumsandproductsforrecurringsequences, TheFibonaciQuaterly, 13(2) (1975), 115-120.

[9]. J. Mc. Laughlin, Combinatorialidentitiesderivingfromthenthpower of a 2x2 matrix, Integers: Electronic J. of CombinatorialNumberTheory 4(2004), 1-15.

[10]. J. McLaughlin, B. Sury: Powers of matrixandcombinatorialidentities. Electronic Journal of CombinatorialNumberTheory, A13, 5 (2005).

[11]. H. Belbachir, Linearrecurrentsequencesand Powers of a squarematrix , Integers: Electronic J. of CombinatorialNumberTheory, 2006, 1-17.

[12]. Z. Akyuz, S. Halici, On SomeCombinatorialIdentitiesInvolvingtheTerms of GeneralizedFibonacciandLucasSequences, Hacettepe Journal of MathematicsandStatistics, 42(4), 2013, 431-435.

[13]. S. Halici, Z. Akyuz, FibonacciandLucasSequences at NegativeIndices,KonuralpJournal of Mathemtics,4(1), 2016, 172-178.

[14]. A. Daşdemir, Mersenne, JacobsthalandJacobsthalLucasNumberswith NegativeSubscrits, Acta. Math