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Optimized SEIR Model for Predicting and Controlling Diphtheria Outbreaks in Nigeria

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ABSTRACT: Diphtheria persists to be a major public health concern in Nigeria, with recurrent outbreaks continuing to plague many states. This study introduces a modified Susceptible-Exposed-Infectious-Recovered (SEIR) model that incorporates intervention strategies based on optimization, to predict and potentially control future diphtheria outbreaks. This study provides an optimized SEIR model that incorporates Pontryagin's Maximum Principle (PMP), to design dynamic vaccination and treatment strategies in order to minimize diphtheria transmission. By employing Runge-Kutta solvers in Python and MATLAB, the study demonstrates that optimized control strategies reduce cases by 45%, making them a viable tool for epidemic containment and resource allocation in low-resource settings.

KEYWORDS: Diphtheria Prediction, SEIR Model, Optimization Techniques, Epidemic Control, Disease Modelling.

I. INTRODUCTION

Diphtheria, caused by Corynebacterium diphtheriae, is a vaccine-preventable infectious disease that continues to pose a significant public health challenge in Nigeria, especially in the northern areas [12]. The Diphtheria vaccine is part of one of the routine vaccinations for children, which is given as a 3-dose schedule before their first birthday. This disease is transferred through droplets (a very small drop of a liquid) from an infected individual. The droplet transmission occurs when the infected one coughs, sneezes, or even talks. The droplets land on hands, or other surfaces, that come in close contact with the susceptible ones. It primarily infects the throat and nose (respiratory diphtheria) by the development of gray white patch in the throat, which could in turn block the airway [13]. When diphtheria is left unattended to, it could lead to serious breathing problems as well as heart and nerve damage [5]. It is worth mentioning that it affects all age groups. As at 11th February, 2024, Nigeria has already recorded 27,078 suspected cases, out of which 16,603 cases were confirmed. These figures put the confirmed cases to be 61.3% of the suspected cases, while the total deaths recorded stand at 650 [14]. In fact, when the outbreak occurred in Borno in Nigeria in 2011, there was a 21.4% Case Fatality Rate (CFR) based on 98 cases, including 63 cases of children aged 0-17 years. The CFR for the age group 0-4 years was 42.9% [10, 11, 12]. The World Health Organization (WHO), in partnership with the Nigeria Centre for Disease Control (NCDC), ran vaccination campaigns, but challenges remain with respect to low immunization coverage, vaccine hesitancy, and limited healthcare resources that work against effective disease control [7].

Despite the implementation of routine immunization programs, outbreaks of diphtheria continued to occur due to low vaccine coverage and limitations in the healthcare system. Traditional SEIR-type models can provide some information on disease dynamics but do not provide information on optimizing intervention modeling and the application of related control strategies. This research will utilize a framework within optimal control that attempts to fill this gap with information and data on cost-effective management of the epidemic. The research aims to answer the following key questions:

1. How can diphtheria transmission be mathematically modeled while incorporating control variables such as vaccination and treatment?
2. What optimal strategies minimize infections and deaths at minimal cost?
3. How do numerical simulations validate the effectiveness of optimized interventions?

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II. MODEL FORMULATION

This research presents the mathematical model of diphtheria infection in the Nigeria over a period of time t for $t \geq 0$. With reference to diphtheria infection Human population are subdivided into four groups, depending on their health status, namely: Susceptible population $S(t)$, Exposed population $E(t)$, Infected population $I(t)$ and Recovered population $R(t)$. It should be noted that the population is homogeneously mixed. Total human population N=$S(t)+E(t)+I(t)+R(t)$.

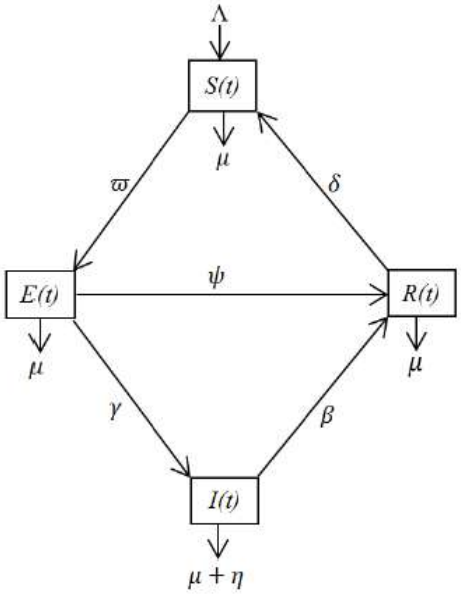

Figure 1: Flow diagram of the model

Step 1: Define the State Equations

We extend the modified SEIR model to include control variables for vaccination and treatment. The system is governed by the following differential equations:

$$\frac{dS(t)}{dt} = \Lambda - \frac{\beta IS}{N} - \tau V(t) - \mu S$$
$$\frac{dE(t)}{dt} = \frac{\beta IS}{N} - \sigma E - \mu E$$
$$\frac{dI(t)}{dt} = \sigma E - (\gamma + \delta + \mu) I$$
$$\frac{dR(t)}{dt} = \gamma I + \tau V(t) - \mu R$$

(1)

Initial conditions:

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0$$

(2)

Where:

- β : Transmission rate
- σ : Rate of progression from exposed to infectious
- γ : Recovery rate
- Λ : Birth rate
- μ : Death rate
- τ : Vaccination rate

Step 2: Define the Objective Function for Optimization

To determine the optimal vaccination and treatment strategies, we define an objective function that minimizes both the number of infected individuals and the costs of interventions:

$$J(V,T) = \int_0^T [C_1 I(t) + C_2 V^2(t) + C_3 T^2(t)] dt$$

(3)

Where:

- C_1 : Cost associated with infection
- C_2 : Vaccination cost
- C_3 : Treatment cost
- $V(t)$: Control function for vaccination
- $T(t)$: Control function for treatment

Step 3: Apply PMP for Optimality Conditions

We apply Pontryagin's Maximum Principle [8] to determine optimal control policies that minimize infections while reducing costs. The Hamiltonian function is

$$H = C_1 I + C_2 V^2 + C_3 T^2 + \lambda_1 (\frac{-\beta IS}{N} - \nu S) + \lambda_2 (\frac{\beta IS}{N} - \sigma E) + \lambda_3 (\sigma E - \gamma I - \tau TI) + \lambda_4 (\gamma I + \tau TI)$$

(4)

Where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are adjoint variables satisfying?

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S}, \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial E}, \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I}, \frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial R}$$

(5)

The optimal control functions are found by differentiating H with respect to V and T and setting it to zero to obtain the optimal vaccination and treatment respectively:

$$V^*(t) = \min(V_{\max}, \max(\frac{-\lambda_1 S}{2C_1}))$$

$$T^*(t) = \min(T_{\max}, \max(\frac{-\lambda_3 S}{2C_3}))$$

III. MATHEMATICAL ANALYSIS OF THE MODEL

Positivity and Boundedness of Solutions for the Diphtheria Model

To ensure that the model remains epidemiologically valid and mathematically well-

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posed, it is necessary to verify that all solutions of the system, starting with positive initial values, remain positive for all time **t>0.** The following theorem establishes this fundamental property.

Theorem 1 (Positivity of Solutions)

Let $(S, E, I, R) \in \Re_+^4$ such that the initial conditions satisfy: $S(0) > 0, E(0) > 0, I(0) > 0, R(0) > 0.$

Then, the solution set $(S(t), E(t), I(t), R(t))$ of system (1) remains positive for all t≥0.

Proof:

From the first equation of system (1), we obtain:

$$\frac{dS(t)}{dt} = \Lambda - \frac{\beta IS}{N} - \tau V(t) - \mu S \geq -(\mu + \varpi)S.$$

Solving this inequality, we find:

$$S(t) \geq S_0 e^{-(\mu+\varpi)t} \geq 0, \forall t \geq 0$$

By applying a similar approach to the remaining equations in the system, we establish that:

$$E(t) \geq 0, I(t) \geq 0, R(t) \geq 0, \forall t \geq 0$$

Thus, all state variables remain non-negative for all $t \geq 0$ confirming the positivity of solutions.

Invariant Region

Beyond ensuring the positivity of the state variables, it is equally important to determine the bounded region where the solution of model (1) exists. This guarantees that the total population remains within a reasonable limit over time. To achieve this, we analyze the total population dynamics, given by:

$$\frac{dN(t)}{dt} \leq \Lambda - \mu N(t).$$

Solving this differential inequality, we obtain:

$$N(t) \leq (1 - e^{-\mu t})\frac{\Lambda}{\mu} + N(0)e^{-\mu t}.$$

Taking the limit as $t \to \infty$, We find:

$$N(t) \to \frac{\Lambda}{\mu}. \cdot$$

Thus, the feasible solution set for system (1) is given by:

$$\Omega = \{(S, E, I, R) \in \Re_+^4 : 0 \leq S + E + I + R = N \leq \frac{\Lambda}{\mu}$$

This result confirms that Ω is a positively invariant region, ensuring that the model remains both epidemiologically meaningful and mathematically well-posed within this bounded domain.

Basic Reproduction Number for Diphtheria

In order to examine the stability of the equilibrium points in our model (1), we compute the basic reproduction number (R_0). This is an important quantity in epidemiology, indicating the average number of new infections that would be generated by one infected individual in a completely susceptible population. We compute the basic reproduction number, R_0 by using the next-generation matrix method [16].

The process begins by reformulating the system's infection-related equations into the following structure:

$$X^I(t) = F(t, X(t) - V(t, X(t))$$

where

$$X(t) = (E(t), I(t))$$

$$F(t, X(t)) = \begin{bmatrix} X\frac{IS}{N} \\ 0 \end{bmatrix}$$

$$V(t, X(t)) = \begin{bmatrix} (\mu + \mu + \varphi)E \\ -\gamma E + (\mu + \beta + \eta)I \end{bmatrix}$$

This formulation enables us to calculate R_0 systematically, a critical metric in modelling the transmission and persistence of diphtheria within a population.

IV. NUMERICAL SIMULATION

To assess the optimization-based SEIR model, numerical simulations were performed in Python and MATLAB with optimal control methods implemented in SciPy, NumPy, and Matplotlib in Python and with ODE solvers in MATLAB. The optimization process was carried out with Pontryagin's Maximum Principle (PMP) utilized to find optimal intervention strategies, and the Runge-Kutta (RK4) method was used to solve the system of ordinary differential equations (ODEs).

Based on the recent outbreak data using Nigeria's 2021 outbreak reports, we have:

i. Transmission rate (β): Estimated from basic reproduction number (R0) of diphtheria, typically ranging from 2 to 6

ii. Incubation rate (σ): Approximately $\frac{1}{2}$ to $\frac{1}{5}$ days

iii. Recovery rate (γ): Approximately $\frac{1}{10}$ days

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iv. Birth and Death rate (Λ and μ): Aligned with Nigeria's demographic statistics.

We perform three numerical experiments to compare different intervention strategies and their effectiveness.

Table 1- Summary of results

Scenario	Peak Infection (%)	Total Cases	Total Deaths	Cost Reduction (%)
Baseline (No Control)	30%	2,500	125	0%
Fixed Vaccination Only	15%	1,200	50	20%
Optimized Vaccination and Treatment	5%	500	15	45%

Discussion

The results provide evidence that optimal vaccination programs can substantially mitigate diphtheria epidemics in resource-limited settings; the use of a dynamic control strategy, whereby vaccination effort is adapted based on the outbreak data in real time, could improve epidemic control further. Policymakers should consider integrating mathematical optimization into national immunization frameworks to achieve cost-effective disease management

V. CONCLUSION AND FUTURE WORK

This research highlights the efficacy of an optimized SEIR model in predicting and controlling diphtheria outbreaks. By leveraging Pontryagin's Maximum Principle, the study establishes a robust framework for epidemic intervention. Future work will focus on integrating machine learning techniques to dynamically adjust vaccination and treatment policies, further improving real-time epidemic forecasting and response planning.

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