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RP-182: Formulation of Solutions of Standard Bi-Quadratic Congruence modulo Double of Four Raised To the Power N

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ABSTRACT
Here in this current paper, the standard bi-quadratic congruence of composite modulus is studied and formulated for its incongruent solutions. It is found that the said congruence has three types of solutions discussed in three different cases. In firstcase it has exactly eight incongruent solutions; in the second case, it has exactly thirty two incongruent solutions while in the third case, the congruence has exactly sixty-four incongruent solutions. The formulation is tested and verified true by solving some suitable numerical examples. Formulation is the merit of the current paper.

KEY-WORDS: Bi-quadratic congruence, Binomial expansion, Composite modulus, Formulation.

I. INTRODUCTION

Congruence is studied in book of Number Theory. Most of the books on number theory discuss only quadratic congruence of prime and composite modulus. But nothing is seen about bi-quadratic congruence [1], [2], [3]. The author has taken a bold attempt to study standard bi-quadratic congruence of prime and composite modulus. In this regard, the author already has formulated many standard bi-quadratic congruence. Here is one more such congruence formulated earlier in a different cases by the author.

PROBLEM-STATEMENT

Here the problem statement is as under-

" To formulate the solutions of the standard bi − quadratic confrence of the type:
$x^4 \equiv a^4 \pmod{2.4^n}$ in three cases.
Case-I: when a is Let a be an odd positive integer;
Case-II: when $a \not\equiv 0 \pmod 4$, be an even positive integer;
Case-III: when $a = 4l$, l any positive integer".

II. LITERATURE REVIEW

The standard bi-quadratic congruence of prime and composite modulus (seems) found no place in the big ocean of Literature of mathematics. Actually this part of modular arithmetic is kept untouched by the researchers of mathematics. Only the present author has shown his keen interest in this field.

The author already has formulated and got published some standard bi-quadratic congruence of prime and composite modulus in different international journals [4], [5], [6], [7], [8], [9].

III. ANALYSIS & RESULTS

Consider the said congruence: $x^4 \equiv a^4 \pmod{2.4^n}$.

Case-I: Let a be an odd positive integer.

For the solutions, let $x \equiv 2.4^{n-1}k \pm a \pmod{2.4^n}$.

Then, by binomial expansion,

$$x^4 \equiv (2.4^{n-1}k \pm a)^4 \pmod{2.4^n}$$
$$\equiv (2.4^{n-1}k)^4 \pm 4.(2.4^{n-1}k)^3.a + \frac{4.3}{2.1}.(2.4^{n-1}k)^2.a^2 \pm \frac{4.3.2}{3.2.1}.2.4^{n-1}k.a^3 + a^4 \pmod{2.4^n}$$

$\equiv 2.4^n k\{4^{3n-2}k^3 \pm 4^n k^2 a + 4^{n-1}3ka^2 \pm a^3\} + a^4 \pmod{2.4^n}$

$\equiv 0 + a^4 \pmod{2.4^n}$

$\equiv a^4 \pmod{2.4^n}$.

So, $x \equiv 2.4^{n-1}k \pm a \pmod{2.4^n}$ satisfies the said congruence and is a solutions formula.

But for $k = 4$, the formula becomes: $x \equiv 2.4^{n-1}.4 \pm a \ (mod\ 2.4^n)$
$$\equiv 2.4^n \pm a \ (mod\ 2.4^n)$$
$$\equiv 0 \pm a \ (mod\ 2.4^n)$$
This is the same solution as for $k = 0$.
Similarly, for $k = 5 = 4 + 1$,the formula becomes: $x \equiv 2.4^{n-1}.(4+1) \pm a \ (mod\ 2.4^n)$
$$\equiv 0 + 2.4^{n-1} \pm a \ (mod\ 2.4^n)$$
$$\equiv 2.4^{n-1} \pm a \ (mod\ 2.4^n)$$
This is the same solution as for $k = 1$.
Therefore, all the solutions are given by
$$x \equiv 2.4^{n-1}k \pm a \ (mod\ 2.4^n); k = 0, 1, 2, 3.$$
This gives exactly eight incongruent solutions of the congruence if a is an odd positive integer.

Case-II: Let $a \not\equiv 0 \ (mod\ 4)$, be an even positive integer.
For the solutions, let $x \equiv 2.4^{n-2}k \pm a \ (mod\ 2.4^n)$.
Then, by binomial expansion,
$$x^4 \equiv (2.4^{n-2}k \pm a)^4 \ (mod\ 2.4^n)$$
$$\equiv (2.4^{n-2}k)^4 \pm 4.(2.4^{n-2}k)^3.a + \frac{4.3}{2.1}.(2.4^{n-2}k)^2.a^2 \pm \frac{4.3.2}{3.2.1}.2.4^{n-2}k.a^3 + a^4 \ (mod\ 2.4^n)$$
$$\equiv 2.4^n k\{4^{3n-2}k^3 \pm 4^n k^2 a + 4^{n-1}3ka^2 \pm a^3\} + a^4 \ (mod\ 2.4^n)$$
$$\equiv 0 + a^4 \ (mod\ 2.4^n)$$
$$\equiv a^4 \ (mod\ 2.4^n).$$
So, $x \equiv 2.4^{n-2}k \pm a \ (mod\ 2.4^n)$ satisfies the said congruence and is a solutions formula.
But for $k = 16 = 4^2$, the formula becomes: $x \equiv 2.4^{n-2}.4^2 \pm a \ (mod\ 2.4^n)$
$$\equiv 2.4^n \pm a \ (mod\ 2.4^n)$$
$$\equiv 0 \pm a \ (mod\ 2.4^n)$$
This is the same solution as for $k = 0$.
Similarly, for $k = 17 = 4^2 + 1$,the formula becomes: $x \equiv 2.4^{n-2}.(4^2+1) \pm a \ (mod\ 2.4^n)$
$$\equiv 0 + 2.4^{n-2} \pm a \ (mod\ 2.4^n)$$
$$\equiv 2.4^{n-2} \pm a \ (mod\ 2.4^n)$$
This is the same solution as for $k = 1$.
Therefore, all the solutions are given by
$$x \equiv 2.4^{n-2}k \pm a \ (mod\ 2.4^n); k = 0, 1, 2, 3, \dots\dots\dots\dots..15.$$
This gives exactly thirty two incongruent solutions of the congruence if a is an even positive integer not divisible by 4.

Case-III: Let $a = 4l, l \ any\ positive\ integer$.
For the solutions, let $x \equiv 2.4^{n-3}k + 4l \ (mod\ 2.4^n)$.
Then, by binomial expansion,
$$x^4 \equiv (2.4^{n-3}k + 4l)^4 \ (mod\ 2.4^n)$$
$$\equiv (2.4^{n-3}k)^4 + 4.(2.4^{n-3}k)^3.4l + \frac{4.3}{2.1}.(2.4^{n-3}k)^2.(4l)^2 + \frac{4.3.2}{3.2.1}.2.4^{n-3}k.(4l)^3 + (4l)^4 (mod\ 2.4^n)$$
$$\equiv 2.4^n k\{4^{3n-2}k^3 + 4^n k^2.4l + 4^{n-1}3k(4l)^2 + (4l)^3\} + (4l)^4 \ (mod\ 2.4^n)$$
$$\equiv 0 + (4l)^4 \ (mod\ 2.4^n)$$
$$\equiv (4l)^4 \ (mod\ 2.4^n).$$
So, $x \equiv 2.4^{n-3}k + 4l \ (mod\ 2.4^n)$ satisfies the said congruence and is a solutions formula.
But for $k = 64 = 4^3$, the formula becomes: $x \equiv 2.4^{n-3}.4^3 + 4l \ (mod\ 2.4^n)$
$$\equiv 2.4^n + 4l \ (mod\ 2.4^n)$$
$$\equiv 0 + 4l \ (mod\ 2.4^n)$$
This is the same solution as for $k = 0$.
Similarly, for $k = 65 = 4^3 + 1$,the formula becomes:
$$x \equiv 2.4^{n-3}.(4^3+1) + a \ (mod\ 2.4^n)$$
$$\equiv 0 + 2.4^{n-3} + 4l \ (mod\ 2.4^n)$$
$$\equiv 2.4^{n-3} + 4l \ (mod\ 2.4^n)$$
This is the same solution as for $k = 1$.
Therefore, all the solutions are given by
$$x \equiv 2.4^{n-3}k + 4l \ (mod\ 2.4^n); k = 0, 1, 2, 3, \dots\dots\dots\dots..63.$$
This gives exactly sixty-four incongruent solutions of the congruence if a is an even positive integer not divisible by 4.

IV. ILLUSTRATIONS

Example-1: Consider $x^4 \equiv 81 \ (mod\ 512)$.
It can be written as $x^4 \equiv 3^4 \ (mod\ 2.4^4)$
It is of the type $x^4 \equiv a^4 (mod\ 2.4^n)\ with\ n = 4, a = 3, an\ odd\ positive\ integer$.
It has exactly eight incongruent solutions.
The solutions are given by
$$x \equiv 2.4^{n-1}k \pm a \ (mod\ 2.4^n)\ with\ n = 4.$$
$$\equiv 2.4^3k \pm 3 \ (mod\ 2.4^4)$$
$$\equiv 128k \pm 3 \ (mod\ 512); k = 0, 1, 2, 3.$$
$$\equiv 0 \pm 3; 128 \pm 3; 256 \pm 3; 384 \pm 3 \ (mod\ 512)$$
$$\equiv 3, 509; 125, 131; 253, 259, 381, 387 \ (mod\ 512).$$
Example-2: Consider $x^4 \equiv 2401 \ (mod\ 2048)$.
It can be written as $x^4 \equiv 7^4 \ (mod\ 2.4^5)$
It is of the type $x^4 \equiv a^4 (mod\ 2.4^n)\ with\ n = 5, a = 7, an\ odd\ integer$.
It has exactly eight incongruent solutions.
The solutions are given by
$$x \equiv 2.4^{n-1}k \pm a \ (mod\ 2.4^n)\ with\ n = 5.$$
$$\equiv 2.4^4k \pm 7 \ (mod\ 2.4^5)$$
$$\equiv 512k \pm 7 \ (mod\ 2048); k = 0, 1, 2, 3.$$
$$\equiv 0 \pm 7; 512 \pm 7; 1024 \pm 7; 1536 \pm 7 \ (mod\ 2048)$$

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$$\equiv 7, 2041; 505, 519; 1017, 1031; 1529, 1543 \ (mod\ 2048).$$

Example-3: Consider $x^4 \equiv 1296\ (mod\ 2048)$.
It can be written as $x^4 \equiv 6^4\ (mod\ 2.4^5)$
It is of the type $x^4 \equiv a^4 (mod\ 2.4^n)\ with\ n = 5, a = 6 \not\equiv 0 (mod\ 4), an\ even\ integer$.
It has exactly Thirty-two incongruent solutions.
The solutions are given by

$$x \equiv 2.4^{n-2}k \pm a\ (mod\ 2.4^n)\ with\ n = 4, a = 6 \not\equiv 0\ (mod\ 4), an\ even\ integer.$$
$$\equiv 2.4^3 k \pm 6\ (mod\ 2.4^4)$$
$$\equiv 128k \pm 6\ (mod\ 512); k = 0, 1, 2, 3, \dots\dots\dots\dots, 15.$$
$$\equiv 0 \pm 6; 128 \pm 6; 256 \pm 6; 384 \pm 6; \dots\dots\dots\dots; 1920 \pm 6\ (mod\ 2048)$$
$$\equiv 3, 509; 125, 131; 253, 259, 381, 387; \dots\dots\dots\dots; 1914, 1926\ (mod\ 2048).$$

Example-4: Consider $x^4 \equiv 256\ (mod\ 512)$.
It can be written as $x^4 \equiv 4^4\ (mod\ 2.4^4)$
It is of the type $x^4 \equiv a^4 (mod\ 2.4^n)\ with\ n = 4, a = 4 \equiv 0\ (mod\ 4), an\ even\ integer$.
It has exactly sixty- four incongruent solutions.
The solutions are given by

$$x \equiv 2.4^{n-3}k + a\ (mod\ 2.4^n)\ with\ n = 4.$$
$$\equiv 2.4^1 k + 4\ (mod\ 2.4^4)$$
$$\equiv 8k + 4\ (mod\ 512); k = 0, 1, 2, 3 \dots\dots\dots, 63.$$
$$\equiv 0 + 4; 8 + 4; 16 + 4; 24 + 4; 32 + 4; \dots\dots\dots\dots, 504 + 4\ (mod\ 512)$$
$$\equiv 4, 12, 20, 28, 36, \dots\dots\dots\dots, 508\ (mod\ 512).$$

Example-5: Consider $x^4 \equiv 4096\ (mod\ 8192)$.
It can be written as $x^4 \equiv 8^4\ (mod\ 2.4^6)$
It is of the type $x^4 \equiv a^4 (mod\ 2.4^n)$ with $n = 6$.
It has exactly sixty-four incongruent solutions.
The solutions are given by

$$x \equiv 2.4^{n-3}k \pm a\ (mod\ 2.4^n)\ with\ n = 4, a = 8 \equiv 0\ (mod\ 4).$$
$$\equiv 2.4^3 k \pm 8\ (mod\ 2.4^6)$$
$$\equiv 128k \pm 8\ (mod\ 8192); k = 0, 1, 2, 3 \dots\dots\dots .63.$$
$$\equiv 0 \pm 8; 128 \pm 8; 256 \pm 8; 384 \pm 8; \dots\dots\dots\dots; 8064 \pm 8\ (mod\ 8192)$$
$$\equiv 3, 509; 125, 131; 253, 259; 381, 387; \dots\dots\dots; 8056, 8072\ (mod\ 8192).$$

V. CONCLUSIONS

Therefore, it is then concluded that the standard bi-quadratic congruence of composite modulus modulo double of four to the power n: $x^4 \equiv a^4\ (mod\ 2.4^n)$ is formulated and
Has three types of solutions. It has exactly eight incongruent solutions given by

$$x \equiv 2.4^{n-1}k \pm a\ (mod\ 2.4^n), k = 0, 1, 2, 3;\ \text{if a is an oddpositive integer.}$$

Also, it the congruence has thirty-two incongruent solutions given by

$$x \equiv 2.4^{n-2}k \pm a\ (mod\ 2.4^n), k = 0, 1, 2, 3, \dots\dots\dots\dots, 15;\ \text{if a} \not\equiv 0\ (mod\ 4)\ \text{and an even positive integer.}$$

But it has exactly sixty-four incongruent solutions given by

$$x \equiv 2.4^{n-3}k + 4\ (mod\ 2.4^n), k = 0, 1, 2, 3, \dots\dots\dots 63;\ \text{if a is integer multiple of 4.}$$

REFERENCES

[1]. Thomas Koshy, 2009, Elementary Number Theory with Applications, Academic Press, Second Edition, Indian print, New Dehli, India, ISBN: 978-81-312-1859-4.

[2]. Zuckerman at el, An Introduction to The Theory of Numbers, fifth edition, Wiley India (P) Ltd, 2008.

[3]. Burton David M., Elementary Number Theory, seventh edition, Mc Graw Hill education (India), 2017.

[4]. Roy B M,Formulation of solutions of some classes of standard bi-quadratic congruence of composite modulus, International Journal of Engineering Technology Research and Management(IJETRM), ISSN:2456-9348, Vol-3, Issue-2, Feb-19.

[5]. Roy B M, Formulation of Some Classes of Solvable Standard Bi-quadratic Congruence of Prime-power Modulus, (IJSRED), ISSN: 2581-7175, Vol-02, Issue-01, Jan-Feb-19.

[6]. Roy B M,An Algorithmic Method of Finding Solutions of Standard Bi-quadratic Congruence of Prime Modulus, (IJSDR), ISSN: 2455-2631,Vol-04, Issue-04, April-19.

[7]. Roy B M,Formulation of solutions of standard biquadratic congruence of even composite modulus,International Journal of Engineering Technology Research and Management(IJETRM), ISSN:2456-9348, Vol-3, Issue-11, Nov-19.

[8]. Roy B M, Formulation Solutions of a special standard bi-quadratic congruence- modulo a powered odd prime, International Journal of Engineering Technology Research and Management (IJETRM), ISSN:2456-9348 , Vol-05, Issue-04, April-21.

[9]. Roy B M, Solving standard bi-quadratic congruence modulo a product of powered odd prime & powered four, International Journal of Engineering Technology Research and Management (IJETRM), ISSN:2456-9348 , Vol-05, Issue-07, July-21.