International Journal of Advances in Engineering and Management (IJAEM)
Volume 5, Issue 3 March 2023, pp: 1263-1267 www.ijaem.net ISSN: 2395-5252

Z – Matrices and Inverse Z – Matrices

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Date of Submission: 12-03-2023 Date of Acceptance: 22-03-2023

ABSTRACT

We consider Z-matrices and inverse Z-matrices, i.e. those nonsingular matrices whose inverses are Z-matrices. Recently Fiedler and Markham introduced a classification of Z-matrices. This classification directly leads to a classification of inverse Z-matrices. Among all classes of Z-matrices and inverse Z-matrices, the classes of M-matrices, N0-matrices, F0-matrices, and inverse M-matrices, inverse N0-matrices and inverse F0-matrices, respectively, have been studied in detail. Here we discuss each single class of Z-matrices and inverse Z-matrices as well as considering the whole classes of Z-matrices and inverse Z-matrices.

CHARACTERIZING Z – MATRICES

In this section, we have given the basic results , lemma and the theorems of Z - matrices .

Definition: 1.1

Suppose $0 \neq x \in R^n$ and $y \in R^n$. Let P be the permutation matrix chosen so that

$$P_X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$ In which $X_1 > 0$, $X_2 > 0$

and $X_3 = 0$ (entry - wise) and suppose that

$$P_Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

is partitioned conformally with x . Except for X_1 and X_2 any one or two of X_1 , X_2 and X_3 may be empty .

Consider the following properties :

(P_1) If X_1 is empty and X_3 is not , then $Y_3 \geq 0$ and if X_2 is empty and X_3 is not , then $Y_3 \leq 0$

(P_2) $X_1 \circ Y_1 \nleq 0$ and $X_2 \circ Y_2 \nleq 0$

(P_3) $X_1 \circ Y_1 \nless 0$ and $X_2 \circ Y_2 \nless 0$

LEMMA : 1.2

Let $A \in M_n(R)$. Then $A \in Z$ if and only if for each $0 \neq x \in R^n$.
x and Ax satisfy (P_1) .

PROOF :

Let $A \in M_n(R)$.
If n = 1 , the result is clear .
Assume hence forth then $n \geq 2$.
In order to prove necessity , assume that $A \in Z$ and

$$0 \neq x = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \in R^n .$$

in which $X_1 > 0$, $X_2 < 0$ and $X_3 = 0$. If X_1 is empty and X_3 is not , then , partitioning y and A conformally with x , we have

$$Y = \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} = AX$$

[since $x \neq 0$, X_2 is non empty] .
Thus ,

$Y_3 = A_{32}$, $X_2 \geq 0$.

Similarly ,
if X_2 is empty and X_3 is not , if follows that $Y_3 \leq 0$.

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Thus ,
x and Ax satisfy by (P_1) .
Conversely ,
Assume the contrary , say $a_{ij} > 0$ some $i \neq j$.
If $x = - e_j$, then X_1 is empty and X_3 is not .
But $Y_3 \ngeq 0$, which contradicts (P_1) .
Hence the lemma .

THEOREM : 1.3

Let $A \in M_n(R)$. Then $A \in K_0$ if and only if for each $0 < x \in R^n$, x and Ax are doubly closed - sign related .

PROOF:

Let $A \in M_n(R)$.

Since the result is clear for n = 1 , we assume that $n \geq 2$.

Assume that for every $0 \neq x \in R^n$. x and Ax are doubly closed - sign related .

Let $0 \neq x \in R^n$.

Then x and Ax satisfy (P_1) and (P_3) .
Hence , applying the lemma (2.2), $A \in Z$. Further , since $x \neq 0$, (P_3) implies that there is a subscribt i such that $x_i \neq 0$ and $x_i y_i \geq 0$.
This , in turn , implies that A is an M - matrix .
Conversely ,

Suppose that A is an M - matrix , $0 \neq x \in R^n$ and y = Ax .

Then , $A \in Z$ and it follows from the lemma that (P_1) holds . Writing x in partitioned form and partitioning y and A conformally with x , we have

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

in which $X_1 > 0$, $X_2 < 0$ and $X_1 = 0$. Thus assuming X_1 is non empty ,
we have $Y_1 = A_{11} X_1 + A_{12} X_2$ or $A_{11} X_1 = Y_1 - A_{12} X_2$.

Now if $Y_1 < 0$, it follows that $A_{11} X_1 < 0$.

Thus , there is $\in > 0$ such that $(A_{11} + \in I) X_1 < 0$.

But this contradicts the fact that $A_{11} + \in I$ is a non singular
M - matrix and hence a P – matrix .

Thus , $Y_1 = \nless 0$. So (P_3) holds .
Hence the theorem .

THEOREM : 1.4

Let $A \in M_n(R)$. Then $A \in N_0$ if and only if

(i) for each $0 < x \in R^{n-1}$ and every $i \in N$, x and Ax are doubly closed - sign related and
(ii) there is a vector $0 < \hat{x} \in R^n$ such that $A \hat{x}$.

PROOF :

Let $A \in M_n(R)$. if $A \in N_0$, then each proper submatrix is an H - matrix and it follows from theorem 2.3 that (i) holds .

Now A = t I – B in which $B \geq 0$ is irreducible and $\rho_{n-1}(B) \leq t \leq \rho(B)$.
There is a positive vector $\hat{x} \in R^n$ such that $B \hat{x} = \rho^{(B)} \hat{x}$.
Thus , $A \hat{x} = (t I - \rho(B)) \hat{x} < 0$.
Conversely ,

Suppose that A satisfies (i) and (ii) , say $A \hat{x} = y < 0$. Then there is $\in > 0$ such that

$$(A + \in I) \hat{x} = Z < 0 .$$

Since (i) holds , $A \in Z$ and it follows from theorem (2.3) that principal submatrices of order n – 1 are M - matrices .
Thus , A is either an M - matrix or an N_0 matrix .
If A is an M - matrix , then $A + \in I$ is non - singular M - matrix and hence $(A + \in I)^{-1} \geq 0$.
Hence , $\hat{x} = (A + \in I)^{-1} Z < 0$, a contradiction .
Thus , A is an N_0 matrix .
Hence the proof .

THEOREM : 1.5

Let $A \in M_n(R)$ in which $n \geq 3$. For k = 1 , 2 , … , n^{-2}, $A \in L_k$ if and only if

(i) for each $0 \neq x \in R^k$ and every $\alpha \subseteq N$ with $|\alpha| = k$, x and $A[\alpha]$ x are doubly closed sign - related and

(ii) there is $\beta \subseteq N$ with $|\beta| = k + 1$ and a vector $0 < \hat{x} \in R^{k+1}$ such that

$$A[\beta] \hat{x} < 0 .$$

PROOF :
Let $A \in M_n(R)$ and $k \in \{1, 2, \ldots, n - 2\}$.

If $A \in L_k$, then each proper principal submatrix of order k is an M - matrix and it follows from theorem (2.3) that (i) holds .
Further there is $\beta \subseteq N$ with $|\beta| = k + 1$ such that $A[\beta]$ is an N_0 - matrix , say $A[\beta] = t I - B = \rho(B)$.
Let $\hat{x}$ be the Perron vector associated with $\rho(B)$.
Then , $A[\beta] \hat{x} = (t - \rho(B)) \hat{x} < 0$.
Conversely ,

Suppose that A satisfies (i) and (ii) for some $k \in \{1, 2, \ldots, n - 2\}$ say $A[\beta] \hat{x} = Y < 0$ in which is $\beta \subseteq N$ with $|\beta| = k + 1$.

Since (i) holds and , $A \in Z$ and it follows from theorem (2.3) that all principal submatrices of order k or less are M - matrices .

Then it follows from theorem (2.4) that $A [\beta]$ is an N_0 - matrix .

Thus , $A \in L_k$.

Hence the proof .

THEOREM : 1.6

Let $A \in M_n (R)$ .Then Let $A \in L_0$ if and only if

(i) for every $0 \neq x \in R^n$, x and Ax satisfy (P_1) and

(ii) for every $k \in N$, there is $0 \neq x \in R^n$ .and $\alpha \subseteq N$ with $| \alpha | = k$ such that $x \circ A [\alpha] x \leq 0$.

PROOF :

Let $A \in M_n (R)$ and $K \in N$.

If $A \in L_0$ then $A \in Z$ and (i) follows from the lemma .

Further A has atleast one negative diagonal entry , say a_{ij} .

Then , for each $\alpha \subseteq N$ with $| \alpha | = k$ and $j \in \alpha$, $A [\alpha] e_j \leq 0$, where e_j denote the j^{th} standard basis vector .

Thus , $e_j \circ A [\alpha] e_j \leq 0$, which establishes necessity .

Conversely ,

Suppose (i) and (ii) holds .

Since (ii) holds , $A \in Z$ and in , light of the theorems (2.3 – 2.5) .

(ii) implies that $A \notin L_k$ for $k \in N$.

Thus , $A \in L_0$, which completes the proof .

Hence the theorem .

Z – MATRICES AND ITS INVERSE

2.1 Z – MATRICES

In this chapter we have 2 sections . In this section we have the Z - matrices and inverse Z – matrices **.**

DEFINITION : 2.1.1

Let L_s (for s = 0 , … , n) denote the class of matrices consisting of real $n \times n$ matrices which have the form

$A = t I – B$, where $B \geq 0$ and $\rho_s (B) \leq t \leq \rho_{s+1}$ ……… (3.1.1)

Here

$\rho_s (B) := \max \{ \rho (B) : B$ is an $s \times s$ principal submatrix of B $\}$ and we get

$\rho_0 (B) := - \infty$ and $\rho_{n+1} (B) := \infty$.

REMARK : 2.1.2

If one considers matrices of different dimensions, one should introduce another index which gives the dimension of the matrices . Hence one should use $L_{s,n}$ in definition (3.1.1) .

Then the two classes $L_{s,n}$ and $L_{t,m}$ consist of matrices of the same type . (eg . M – matrices , N_0 - matrices) if and only if $n – s = m – t$.

THEOREM : 2.1.3

Let $A \in R^{n,n}$ be a nonsingular matrix , and let λ be a real eigen value of A with $\lambda \neq \rho (A)$. Then

$\lambda \leq \rho_{[n / 2]} (A)$ ………………… (3.1.2)

Since Z - matrices are closely related to non negative matrices , we now obtain .

THEOREM : 2.1.4

Let $A \in L_s$. Then

(i) $\det A \geq 0$ if $s = n$,

(ii) $\det A \leq 0$ if $[n / 2] \leq s < n$.

PROOF :

It is well known that (i) holds .

Let $A \in L_s$ with $A = t I – B$, $B \geq 0$and a fixed $t \in R$.

Now consider the characteristic polynomial of B ,

$\chi (z) = \det (z I – B)$.

The real zeros of $\chi (z)$ indicate the changes of the sign of the determinant of Z - matrices corresponding to the matrix B. But the zeros of $\chi (z)$ are the eigen values of B .

Hence , with theorem 3.1.3 (ii) follows .

Hence the theorem .

EXAMPLE : 2.1.5

Let J_k denote the $k \times k$ matrix of all ones . First assume that n is even . Let $r := n / 2$ .Then let $A_{11} := \alpha I_r - J_r$ and $A_{22} := \beta I_{n-r} - J_{n-r}$.

If $s \leq \alpha$, $\beta < s + 1$; then $\alpha I_r - J_r$ and $\beta I_{n-r} - J_{n-r}$ are in L_s .

Now , if $s < n / 2$, then $\det A_{11} < 0$ and $\det A_{22} < 0$.

However , $A = A_{11} \oplus A_{22} \in L_s$ and $\det A > 0$.

Similarly , one can construct examples for an odd dimension n .

Thus , all classes of Z - matrices consisting of matrices of different dimensions , except the classes of M - matrices , N_0 - matrices and F_0 - matrices include matrices with different signs of their determinants.

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However , if we consider the classes L_s which consist by definition of matrices of the same definition n , then det $A \leq 0$ if $A \in L_s$ and $[n / 2] \leq s < n$.
Obviously , every Z - matrix A , except an M - matrix , has at least one negative eigen value .
However , if $A \in L_s$ with $[n / 2] \leq s < n$, then theorem states that A has exactly one negative eigen value .

THEOREM : 2.1.6

Let $A = [a_{ij}]$ be a Z – matrix . let n (A) denote the smallest real eigen value of A . Let $a = \max \{ a_{ij} \}$. Then the circle $c (A) : = \{ z \in \mathbb{C}^n : | z – a | \leq a – n (a) \}$ contains all eigen values of A .

PROOF :

Let λ be an eigen value of A .

Since $a = \max \{ a_{ij} \}$, the matrix $aI – A$ is nonnegative .
Hence , the modules of each eigenvalues of $aI – A$ is less than $\rho (aI – A)$.
Thus ,

$$| a - \lambda | \leq \rho (aI – A) = a – n (A) .$$

Hence the theorem .

2.2 INVERSE Z – MATRICES

In this section , we have definition and the theorems of Inverse Z – matrices **.**

DEFINITION : 2.2.1

A nonsingular matrix is called an inverse Z - matrix if C^{-1} is a Z - matrix . More precisely , a nonsingular matrix $C \in R^{n \cdot n}$ is called inverse L_5 – matrix if $C^{-1} \in L_s$ for one s with $S \in \{ 0 , \ldots , n \}$.

Thus , inverse M - matrices are inverse L_n - matrices , inverse N_0 - matrices are inverse L_{n-1} - matrices and inverse F_0 - matrices are inverse L_{n-2} - matrices .

THEOREM : 2.2.2

Let C be an inverse Z - matrix partitioned as

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Where C_{11} is a non singular principal submatrix of C of orbitrary order . Then C / C_{11} is also an inverse Z - matrix .

PROOF :

Let $A : = C^{-1}$ be partitioned conformally with C so that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Using the explicit form of C^{-1} as in 3.2.2 we have

$$A_{22} = \left(C/C_{11} \right)^{-1} .$$

Hence , A_{22} is nonsingular . But A_{22} is a principal submatrix of a Z – matrix ; therefore A_{22} itself is a Z – matrix .

Thus , $C/C_{11} = A_{22}^{-1}$ is an Z – matrix .

Hence the theorem .

DEFINITION : 2.2.3

$A = [a_{ij}] \in R^{n \cdot n}$ is of type - D if

$$a_{ij} = \begin{cases} a_i , i \leq j \\ a_j , i > j \end{cases} \quad \text{where } a_n > a_{n-1} > \ldots > a_1$$

Markham proved that the inverse of a type D - matrix A , satisfying $a_1 > 0$ is a tridiagonal M - matrix .

THEOREM : 2.2.4

Suppose $A = [a_{ij}] \in R^{n \cdot n}$ is a matrix of type - D with $a_1 \neq 0$. Let s denote the number of nonpositive parameters in the sequence $a_n > \ldots > a_1$. Then A^{-1} is a tridiagonal Z - matrix and $A^{-1} \in L_{s-1}$, where $L_{-1} : = L_n$.

PROOF :

Since $a_1 \neq 0$, A is nonsingular . In the following we prove by induction on the dimension n that the inverse of A is a tridiagonal Z - matrix .
For n = 1 and n = 2 this is obvious .
Now we partition A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Such that $A_{11} \in R^{n \cdot n}$ and $A_{22} \in R^{n-r,\, n-r}$, $1 \leq r < n$, are non singular .
The inverse of A is given by

$$A^{-1} = \begin{bmatrix} \left(A/A_{22} \right)^{-1} & -A^{-1}_{11} A_{12} \left(A/A_{11} \right)^{-1} \\ -A^{-1}_{11} A_{21} \left(A/A_{11} \right)^{-1} & \left(A/A_{11} \right)^{-1} \end{bmatrix}$$

Proposition (3.2.8) and the inductive hypothesis yield that $\left(A/A_{11} \right)^{-1}$ and $\left(A/A_{22} \right)^{-1}$ are tridiagonal Z - matrices .

As seen in 3.2.4 we have $A_{11}^{-1}A_{12} = e_r \xi_{n-r}^T$.

Moreover , since he first diagonal entry of $\left(A/A_{11} \right)$ is a $_{r+1}$ – a $_r$ > 0 , we obtain with (3.2.3) that only the entry at position (r , 1) of $-A_{11}^{-1}A_{12}\left(A/A_{11} \right)^{-1}$ is different from zero and that this entry is negative .

Then , with the symmetry of A , we have that A^{-1} is a tridiagonal Z – matrix .

If s = 0 , A is nonnegative .

Thus , A^{-1} is an M - matrix .

If s ≥ 1 , if follows from (3.2.5) that det A < 0 .

Moreover , all principal minors of order greater than n – s are nonpositive .

However , the determinant of the principal matrix consisting of the rows and columns s + 1 , … , n is positive .

But then theorem gives that $A^{-1} \in L_{s-1}$.

Hence the theorem .

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